Supplemental Material

I. INVERSION OF THE FLUCTUATION PROPAGATOR

Here we invert the matrix fluctuation propagator defined by the left-hand side of Eq. (7):

$$\tilde{\Delta}_1(\varepsilon) - \pi T \sum_{\varepsilon'} \lambda(\varepsilon, \varepsilon') \frac{\partial F_0(\varepsilon')}{\partial \tilde{\Delta}(\varepsilon')} \tilde{\Delta}_1(\varepsilon') = \phi(\varepsilon). \tag{S1}$$

Evaluating the derivative of $F(\varepsilon') = \tilde{\Delta}(\varepsilon')/[\varepsilon'^2 + \tilde{\Delta}^2(\varepsilon')]^{1/2}$, we get

$$\tilde{\Delta}_1(\varepsilon) - \pi T \sum_{\varepsilon'} \lambda(\varepsilon, \varepsilon') F_0(\varepsilon') \frac{\tilde{\Delta}_1(\varepsilon')}{\tilde{\Delta}(\varepsilon')} + \pi T \sum_{\varepsilon'} \frac{\lambda(\varepsilon, \varepsilon') \tilde{\Delta}^2(\varepsilon')}{[\varepsilon'^2 + \tilde{\Delta}^2(\varepsilon')]^{3/2}} \tilde{\Delta}_1(\varepsilon') = \phi(\varepsilon). \tag{S2}$$

Since both $\tilde{\Delta}(\varepsilon')$ and $\tilde{\Delta}_1(\varepsilon')$ are logarithmically slow functions of ε', the second sum can be easily evaluated and we arrive at

$$\tilde{\Delta}_1(\varepsilon) - \pi T \sum_{\varepsilon'} \lambda(\varepsilon, \varepsilon') F_0(\varepsilon') \frac{\tilde{\Delta}_1(\varepsilon')}{\tilde{\Delta}(\varepsilon')} + L_0^{-1} \lambda(\varepsilon, T_c) \tilde{\Delta}_1(T_c) = \phi(\varepsilon), \tag{S3}$$

where L_0 is the fluctuation propagator at zero frequency and momentum in the BCS theory:

$$L_0^{-1}\left(\frac{T}{T_c}\right) = \pi T \sum_{\varepsilon} \frac{\Delta^2(T)}{[\varepsilon^2 + \Delta^2(T)]^{3/2}} = \begin{cases} \dfrac{7\zeta(3)\Delta^2(T)}{4\pi^2 T^2}, & T_c - T \ll T_c; \\ 1, & T \ll T_c. \end{cases} \tag{S4}$$

The value of $\tilde{\Delta}_1(T_c)$ can be easily obtained from Eq. (S3). In order to do this we multiply it by $F(\varepsilon)$ and sum over ε. Using the SCE (4), we immediately see that the first two terms in Eq. (S3) cancel and we obtain

$$\tilde{\Delta}_1(T_c) = \frac{L_0}{\Delta(T)} \pi T \sum_{\varepsilon} F_0(\varepsilon) \phi(\varepsilon), \tag{S5}$$

where we use that, according to Eq. (6), $\tilde{\Delta}(T_c) = \Delta(T)$.

II. CORRELATION FUNCTION $\langle \Delta_1 \Delta_1 \rangle$ DUE TO MESOSCOPIC FLUCTUATIONS

In this Section we evaluate the zero-momentum correlation function

$$\Phi = \frac{\langle \tilde{\Delta}_1(T_c) \tilde{\Delta}_1(T_c) \rangle_{\mathbf{q}=0}}{\tilde{\Delta}(T_c) \tilde{\Delta}(T_c)} \tag{S6}$$

due to mesoscopic fluctuations of F_{dis} and λ_{dis}.

Correlation function $\langle F_{\text{dis}} F_{\text{dis}} \rangle$

The correlator of Gorkov functions in the Matsubara representation is calculated with the help of imaginary-time replica sigma-model following the line of Ref. [S1]. The resulting expression has the form

$$\langle F_{\text{dis}}(\varepsilon, \mathbf{r}) F_{\text{dis}}(\varepsilon', \mathbf{r}') \rangle = F_0(\varepsilon) F_0(\varepsilon') \frac{[\Pi_{\varepsilon\varepsilon'}(\mathbf{r}, \mathbf{r}')]^2}{(\pi\nu)^2}, \tag{S7}$$

where ν is the 2D one-particle DOS at the Fermi level (per single spin projection), and Π is the diffusion operator on top of the superconducting state:

$$\Pi^{-1}_{\varepsilon\varepsilon'} = -D\nabla^2 + \mathfrak{E}(\varepsilon) + \mathfrak{E}(\varepsilon'), \tag{S8}$$

where

$$\mathfrak{E}(\varepsilon) = \sqrt{\varepsilon^2 + \tilde{\Delta}^2(\varepsilon)}. \tag{S9}$$

For the zero Fourier component we get

$$\langle F_{\rm dis}(\varepsilon) F_{\rm dis}(\varepsilon')\rangle_{\mathbf{q}=0} = \frac{F_0(\varepsilon)F_0(\varepsilon')}{4\pi^3\nu^2 D} \frac{1}{\mathfrak{E}(\varepsilon)+\mathfrak{E}(\varepsilon')}. \tag{S10}$$

The corresponding contribution to Eq. (S6) has the form

$$\Phi^{(FF)} = \frac{L_0^2}{\Delta^4(T)} \frac{(2\pi T)^4}{4\pi^3\nu^2 D} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4>0} \lambda(\varepsilon_1,\varepsilon_2)\lambda(\varepsilon_3,\varepsilon_4) \frac{F_0(\varepsilon_1)F_0(\varepsilon_2)F_0(\varepsilon_3)F_0(\varepsilon_4)}{\mathfrak{E}(\varepsilon_2)+\mathfrak{E}(\varepsilon_4)}. \tag{S11}$$

This expression is typical to fluctuation contributions to Φ. Among four energy summations, two are logarithmic involving large energies, $\varepsilon \gg T_c$, while the other two come from $\varepsilon \sim T_c$, where $\tilde{\Delta}(\varepsilon)$ can be approximated by $\Delta(T)$. The latter summations introduce the dimensionless function

$$N\left(\frac{T}{T_c}\right) = 16T^2 \sum_{\varepsilon_1,\varepsilon_2>0} \frac{\Delta(T)}{\mathfrak{E}_1\mathfrak{E}_2(\mathfrak{E}_1+\mathfrak{E}_2)} = \frac{2}{\pi}\int_0^\infty \frac{d\theta}{\cosh\theta} \tanh^2\left[\frac{\Delta(T)}{2T}\cosh\theta\right] = \begin{cases} \dfrac{14\zeta(3)}{2\pi^3}\dfrac{\Delta(T)}{T}, & T_c - T \ll T_c; \\ 1, & T \ll T_c. \end{cases} \tag{S12}$$

Performing summations over ε_2 and ε_4 in Eq. (S11) we get

$$\Phi^{(FF)} = \frac{1}{16\pi\nu^2 D\Delta^3(T)} L_0^2 \left(\frac{T}{T_c}\right) N\left(\frac{T}{T_c}\right) \left(2\pi T \sum_{\varepsilon>0} \lambda(\varepsilon, T_c) F_0(\varepsilon)\right)^2 \tag{S13}$$

Summation is done with the help of the SCE (4), and we obtain finally

$$\Phi^{(FF)} = \frac{1}{16\pi\nu^2 D\Delta(T)} L_0^2 \left(\frac{T}{T_c}\right) N\left(\frac{T}{T_c}\right). \tag{S14}$$

Correlation function $\langle \delta\lambda_{\rm dis}\delta\lambda_{\rm dis}\rangle$

Mesoscopic fluctuations of the return probability which determines $\delta\lambda(\varepsilon,\varepsilon') = -\lambda_g^2 \ln[1/\max(\varepsilon,\varepsilon')\tau]$ have been calculated for the normal state in Ref. [S2]:

$$\langle \lambda_{\rm dis}(\varepsilon_1,\varepsilon_2)\lambda_{\rm dis}(\varepsilon_3,\varepsilon_4)\rangle_{\mathbf{q}=0} = \frac{\delta\lambda(\varepsilon_1,\varepsilon_2)\delta\lambda(\varepsilon_3,\varepsilon_4)}{16\pi^3\nu^2 D} \left(\frac{1}{|\varepsilon_{13}|} + \frac{1}{|\varepsilon_{14}|} + \frac{1}{|\varepsilon_{23}|} + \frac{1}{|\varepsilon_{24}|}\right), \tag{S15}$$

where $\varepsilon_{ij} \equiv \varepsilon_i + \varepsilon_j$.

Generalization to the superconducting case is achieved by replacing $\varepsilon \to \mathfrak{E}(\varepsilon)$. All four terms in Eq. (S15) equally contribute to Φ, and we get

$$\Phi^{(\lambda\lambda)} = \frac{L_0^2}{\Delta^4(T)} \frac{(2\pi T)^4}{4\pi^3\nu^2 D} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4>0} \delta\lambda(\varepsilon_1,\varepsilon_2)\delta\lambda(\varepsilon_3,\varepsilon_4) \frac{F_0(\varepsilon_1)F_0(\varepsilon_2)F_0(\varepsilon_3)F_0(\varepsilon_4)}{\mathfrak{E}(\varepsilon_2)+\mathfrak{E}(\varepsilon_4)}. \tag{S16}$$

Analogously to Eq. (S11)], summations over ε_2 and ε_4 yields $N(T/T_c)$, while summations over ε_1 and ε_3 are converted to a logarithmic integral:

$$\Phi^{(\lambda\lambda)} = \frac{1}{16\pi\nu^2 D\Delta(T)} L_0^2 \left(\frac{T}{T_c}\right) N\left(\frac{T}{T_c}\right) \left(\int_0^{\zeta_*} \delta\lambda(\zeta,\zeta_*) \cosh\lambda_g(\zeta_*-\zeta)\, d\zeta\right)^2, \tag{S17}$$

and hence

$$\Phi^{(\lambda\lambda)} = \frac{1}{16\pi\nu^2 D\Delta(T)} L_0^2 \left(\frac{T}{T_c}\right) N\left(\frac{T}{T_c}\right) [\cosh\lambda_g\zeta_* - 1]^2. \tag{S18}$$

Correlation function $\langle F_{\mathbf{dis}}\delta\lambda_{\mathbf{dis}}\rangle$

The cross term is evaluated analogously:

$$\Phi^{(F\lambda)} = -2\frac{1}{16\pi\nu^2 D\Delta(T)}L_0^2\left(\frac{T}{T_c}\right)N\left(\frac{T}{T_c}\right)[\cosh\lambda_g\zeta_* - 1]. \tag{S19}$$

Resulting expression for $\langle\Delta_1\Delta_1\rangle$

Adding (S14), (S18) and (S19), we obtain

$$\Phi = \frac{1}{16\pi\nu^2 D\Delta(T)}K\Big(\frac{T}{T_c}\Big)\cosh^2\lambda_g\zeta_* = \frac{\pi D}{g(g-g_c)\Delta(T)}K\Big(\frac{T}{T_c}\Big), \tag{S20}$$

where we have introduced $K(T/T_c) = L_0^2(T/T_c)N(T/T_c)$, used the relation $g = 4\pi\nu D$, and employed $\cosh^2\lambda_g\zeta_* = g/(g-g_c)$. With the help of Eq. (S6), one readily obtains Eq. (10) of the main text.

It's worth noting that Eq. (S20) agrees with our previous results [S2]. Indeed, in the limit $T \to T_c$, Eq. (S20) can be simplified with the help of Eqs. (S4) and (S12) as

$$\langle\langle\Delta(T)\Delta(T)\rangle\rangle_{\mathbf{q}=0} = \frac{2T^3}{7\zeta(3)\nu^2 D\Delta^2(T)}\cosh^2\lambda_g\zeta_*. \tag{S21}$$

On the other hand, the same correlator can be obtained from the correlation function of the coefficient α in the Ginzburg-Landau (GL) expansion [S2]:

$$\langle\langle\alpha\alpha\rangle\rangle_{\mathbf{q}=0} = \frac{7\zeta(3)}{8\pi^4 DT}\cosh^2\lambda_g\zeta_* \tag{S22}$$

with the help of the relation

$$\langle\langle\Delta(T)\Delta(T)\rangle\rangle_{\mathbf{q}=0} = \frac{\langle\langle\alpha\alpha\rangle\rangle_{\mathbf{q}=0}}{4\beta^2\Delta^2(T)}, \tag{S23}$$

where $\beta = 7\zeta(3)\nu/(8\pi^2T^2)$ is the nonlinear coefficient in the GL functional. One can easily verify that Eq. (S23) coincides with Eq. (S21).

III. DENSITY OF THE SUBGAP STATES

Introducing $\theta = \pi/2 + i\psi$, we rewrite Eq. (12) as

$$-\xi^2\nabla^2\psi + F(\psi) = -\frac{\delta\Delta(\mathbf{r})\sinh\psi}{\Delta_0}, \tag{S24}$$

where $\xi^2 = D/2\Delta_0$, and

$$F(\psi) = -\frac{E}{\Delta_0}\cosh\psi + \sinh\psi - \eta\sinh\psi\cosh\psi. \tag{S25}$$

At the minigap ($E = E_g$), $\cosh\psi_g = \eta^{-1/3}$. For small deviation from the gap, the function $F(\psi)$ in the vicinity of its maximum can be written as

$$F(\psi) \approx \Omega(\psi-\psi_0) - \rho(\psi-\psi_0)^2, \tag{S26}$$

where

$$\Omega = \sqrt{6}\sqrt[4]{1-\eta^{2/3}}\sqrt{\frac{E_g - E}{\Delta_0}}, \qquad \rho = \frac{3}{2}\eta^{1/3}\sqrt{1-\eta^{2/3}}. \tag{S27}$$

Comparison of the linear in ψ terms in the left-hand side of Eq. (S24) defines the relevant length scale

$$L_E = \frac{\xi}{\sqrt{\Omega}}. \tag{S28}$$

At the mean-field level, ψ is real below the gap. Finite DOS corresponds to appearance of a nonzero $\operatorname{Im}\psi$ due to a large negative fluctuation of $\delta\Delta(\mathbf{r})$. Its probability is given by

$$\mathcal{P} \propto \exp\left(-\frac{1}{2f(0)}\int \delta\Delta^2(\mathbf{r})\, d^d r\right) \equiv e^{-S_d}, \tag{S29}$$

where we have used that the instanton size, L_E, exceeds the correlation length, ξ_0, of the order parameter fluctuations. At the quantitative level, the instanton action S can be estimated as follows. To produce a nonzero DOS at $E < E_g$, the optimal fluctuation of $\delta\Delta(\mathbf{r})$ should have the magnitude of $-(E_g - E)$ and the spacial extent of L_E, which immediately gives the estimate [S3]

$$S_d \sim \frac{\Delta_0^2 \xi^d}{f(0)} \left(\frac{E_g - E}{E_g}\right)^{2-d/4}. \tag{S30}$$

To find the numerical coefficient in Eq. (S30), one has to solve the instanton equation. Measuring coordinates in terms of L_E introduced in Eq. (S28), appropriately rescaling $\psi - \psi_0$, and replacing $\sinh\psi$ in the right-hand side of Eq. (S24) by its value at E_g, we rewrite Eq. (S24) in a universal dimensionless form:

$$-\nabla^2\phi + \phi - \phi^2 = h(\mathbf{r}), \tag{S31}$$

where

$$\delta\Delta(\mathbf{r}) = -\frac{\Delta_0 \Omega^2}{\rho \sinh\psi_g}\, h(\mathbf{r}/L_E). \tag{S32}$$

Minimization of the functional $\int h^2(\mathbf{r})\, d^d r$ leads to the fourth order differential equation for $\phi(\mathbf{r})$:

$$(-\nabla^2 + 1 - 2\phi)(-\nabla^2\phi + \phi - \phi^2) = 0. \tag{S33}$$

The spherically symmetric optimal fluctuation solving (S33) in d dimensions satisfies the second-order differential equation [S4]

$$-\nabla^2_{d-2}\phi + \phi - \phi^2 = 0, \tag{S34}$$

where $\nabla^2_{d-2} \equiv \partial^2/\partial r^2 - (d-3)r^{-1}\partial/\partial r$ is the radial part of the Laplace operator in the $(d-2)$-dimensional space. The instanton is characterized by the number

$$a_d = \int h^2 d^d r = 4\int \frac{\phi'^2}{r^2}\, d^d r = \begin{cases} 48\pi/5, & d = 3, \\ 4.1, & d = 2, \end{cases} \tag{S35}$$

where a_3 follows from the exact solution $\phi_3(r) = (3/2)\cosh^{-2}(r/2)$ [S3], while a_2 is obtained by a numerical solution of Eq. (S34).

Returning to the dimensional variables, we get for the instanton action in the limit $\eta \ll 1$:

$$S_d = \frac{8a_d}{6^{d/4}} \frac{\Delta_0^2 \xi^d}{f(0)} \left(\frac{E_g - E}{E_g}\right)^{2-d/4}. \tag{S36}$$

In the 2D case,

$$S_2 = \frac{4a_2}{\sqrt{6}} \frac{D\Delta_0}{f(0)} \left(\frac{E_g - E}{E_g}\right)^{3/2}, \tag{S37}$$

leading to Eqs. (18) and (19).

IV. ROLE OF A FINITE FILM THICKNESS

In films with a finite thickness $d \lesssim \xi_0$, there exists a contribution to the depairing parameter η in Eq. (13) coming from large wave vectors ($qd \gg 1$), where diffusion is three-dimensional:

$$\eta_{3\mathrm{D}} = \frac{2}{\Delta} \int \frac{f(\mathbf{q})}{Dq^2} \frac{d^3\mathbf{q}}{(2\pi)^3} = \frac{1}{\pi^2 \Delta D} \int_{1/d}^{1/l} f(q)\, dq \tag{S38}$$

[here $\Delta \equiv \Delta(T)$ is the temperature-dependent BCS order parameter]. In this region, Coulomb effects are weak and all complications related with the energy dependence of λ and Δ can be neglected. Equation (8) is then replaced by a simpler expression written for an arbitrary 3D wave vector:

$$\delta\Delta(T, \mathbf{q}) = L_q\Big(\frac{T}{T_c}\Big)\, 2\pi T \sum_{\varepsilon>0} F_{\mathrm{dis}}(\varepsilon, \mathbf{q}), \tag{S39}$$

where $L_q(T/T_c)$ is the BCS fluctuation propagator at finite momentum:

$$L_q^{-1}(T/T_c) = 2\pi T \sum_{\varepsilon>0} \frac{\mathfrak{E}(\varepsilon) Dq^2 + 2\Delta^2(T)}{\mathfrak{E}^2(\varepsilon)[Dq^2 + 2\mathfrak{E}(\varepsilon)]}. \tag{S40}$$

In the limit $q\xi_0 \gg 1$, one recovers the known inverse logarithmic decay of the fluctuation propagator [S3]:

$$L_q(T/T_c) \approx \frac{1}{\ln(Dq^2/T_c)}, \qquad q\xi_0 \gg 1. \tag{S41}$$

With the help of Eq. (S7) the correlation function $f(\mathbf{q}) = \langle \delta\Delta\delta\Delta \rangle_{\mathbf{q}}$ can be written as

$$f(\mathbf{q}) = L_q^2\Big(\frac{T}{T_c}\Big) \left(\frac{T\Delta}{\nu_3}\right)^2 \sum_{\varepsilon_1, \varepsilon_2 > 0} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\Pi_{\varepsilon\varepsilon'}(\mathbf{k}+\mathbf{q}/2)\Pi_{\varepsilon\varepsilon'}(\mathbf{k}-\mathbf{q}/2)}{\mathfrak{E}(\varepsilon_1)\mathfrak{E}(\varepsilon_2)}, \tag{S42}$$

where $\nu_3 = \nu/d$ is the 3D DOS at the Fermi level. Integrating over $\mathbf{k}$ with the help of the Feynman's trick we get

$$f(\mathbf{q}) = \frac{1}{8\pi} L_q^2\Big(\frac{T}{T_c}\Big) \left(\frac{T\Delta}{\nu_3 D}\right)^2 \sum_{\varepsilon_1, \varepsilon_2 > 0} \frac{1}{\mathfrak{E}(\varepsilon_1)\mathfrak{E}(\varepsilon_2)} \int_0^1 \frac{dx}{[x(1-x)q^2 + (\mathfrak{E}(\varepsilon_1) + \mathfrak{E}(\varepsilon_2))/D]^{1/2}}. \tag{S43}$$

In the limit $q\xi_0 \gg 1$, only large $\varepsilon_{1,2} \gg T_c$ with $\mathfrak{E}(\varepsilon) \approx |\varepsilon|$ are important. Replacing summations by integrations and using Eq. (S41) we find

$$f(\mathbf{q}) = \frac{1}{8\pi^3} L_q^2\Big(\frac{T}{T_c}\Big) \left(\frac{\Delta}{\nu_3 D}\right)^2 \int_{T_c}^{Dq^2} \frac{d\varepsilon_1}{\varepsilon_1} \frac{d\varepsilon_2}{\varepsilon_2} \int_0^1 \frac{dx}{[x(1-x)q^2 + (\varepsilon_1 + \varepsilon_2)/D]^{1/2}} \approx \frac{1}{8\pi^2 q} \left(\frac{\Delta}{\nu_3 D}\right)^2. \tag{S44}$$

The obtained $1/q$ dependence of the correlation function $f(q)$ leads in Eq. (S45) to the logarithmic contribution from the region $l \ll r \ll d$:

$$\eta_{3\mathrm{D}} = \frac{\Delta(T)}{8\pi^4 D} \left(\frac{1}{\nu_3 D}\right)^2 \ln\frac{d}{l}, \tag{S45}$$

leading to Eq. (17).

For even thicker films with $d \gg \xi_0$, the 2D contribution (15) is small and Eq. (S45) with d replaced by ξ_0 gives the leading contribution to the depairing rate.

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