Supplemental Material

I. INVERSION OF THE FLUCTUATION PROPAGATOR

Here we invert the matrix fluctuation propagator defined by the left-hand side of Eq. (7):

\[
\tilde{\Delta}_1(\varepsilon) - \pi T \sum_{\varepsilon'} \lambda(\varepsilon, \varepsilon') \frac{\partial F_0(\varepsilon')}{\partial \tilde{\Delta}(\varepsilon')} \tilde{\Delta}_1(\varepsilon') = \phi(\varepsilon).
\]  

(S1)

Evaluating the derivative of \( F(\varepsilon') = \tilde{\Delta}(\varepsilon')/|\varepsilon'^2 + \tilde{\Delta}(\varepsilon')|^1/2 \), we get

\[
\tilde{\Delta}_1(\varepsilon) - \pi T \sum_{\varepsilon'} \lambda(\varepsilon, \varepsilon') F_0(\varepsilon') \frac{\tilde{\Delta}_1(\varepsilon')}{\Delta(\varepsilon')} + \pi T \sum_{\varepsilon'} \frac{\lambda(\varepsilon, \varepsilon') \tilde{\Delta}^2(\varepsilon')}{[\varepsilon'^2 + \tilde{\Delta}(\varepsilon')^1/2]} \tilde{\Delta}_1(\varepsilon') = \phi(\varepsilon).
\]

(S2)

Since both \( \tilde{\Delta}(\varepsilon') \) and \( \tilde{\Delta}_1(\varepsilon') \) are logarithmically slow functions of \( \varepsilon' \), the second sum can be easily evaluated and we arrive at

\[
\tilde{\Delta}_1(\varepsilon) - \pi T \sum_{\varepsilon'} \lambda(\varepsilon, \varepsilon') F_0(\varepsilon') \frac{\tilde{\Delta}_1(\varepsilon')}{\Delta(\varepsilon')} + L_0^{-1} \lambda(\varepsilon, T_c) \tilde{\Delta}_1(T_c) = \phi(\varepsilon),
\]

(S3)

where \( L_0 \) is the fluctuation propagator at zero frequency and momentum in the BCS theory:

\[
L_0^{-1} \left( \frac{T}{T_c} \right) = \pi T \sum_{\varepsilon} \frac{\Delta^2(T)}{[\varepsilon^2 + \Delta^2(T)]^{1/2}} = \begin{cases} \frac{7\zeta(3)\Delta^2(T)}{4\pi^2 T^2}, & T_c - T \ll T_c; \\ 1, & T \ll T_c. \end{cases}
\]

(S4)

The value of \( \tilde{\Delta}_1(T_c) \) can be easily obtained from Eq. (S3). In order to do this we multiply it by \( F(\varepsilon) \) and sum over \( \varepsilon \). Using the SCE (4), we immediately see that the first two terms in Eq. (S3) cancel and we obtain

\[
\tilde{\Delta}_1(T_c) = \frac{L_0}{\Delta(T)} \pi T \sum_{\varepsilon} F_0(\varepsilon) \phi(\varepsilon),
\]

(S5)

where we use that, according to Eq. (6), \( \tilde{\Delta}(T_c) = \Delta(T) \).

II. CORRELATION FUNCTION \( \langle \Delta_1 \Delta_1 \rangle \) DUE TO MESOSCOPIC FLUCTUATIONS

In this Section we evaluate the zero-momentum correlation function

\[
\Phi = \frac{\langle \tilde{\Delta}_1(T_c) \tilde{\Delta}_1(T_c) \rangle_{q=0}}{\Delta(T_c) \Delta(T_c)}
\]

due to mesoscopic fluctuations of \( F_{\text{dis}} \) and \( \lambda_{\text{dis}} \).

Correlation function \( \langle F_{\text{dis}} F_{\text{dis}} \rangle \)

The correlator of Gorkov functions in the Matsubara representation is calculated with the help of imaginary-time replica sigma-model following the line of Ref. [S1]. The resulting expression has the form

\[
\langle F_{\text{dis}}(\varepsilon, r) F_{\text{dis}}(\varepsilon', r') \rangle = F_0(\varepsilon) F_0(\varepsilon') \frac{[\Pi_{\nu}^{-1}(r, r')]^2}{(\pi \nu)^2},
\]

(S7)

where \( \nu \) is the 2D one-particle DOS at the Fermi level (per single spin projection), and \( \Pi \) is the diffusion operator on top of the superconducting state:

\[
\Pi_{\nu}^{-1} = -D \nabla^2 + \mathcal{E}(\varepsilon) + \mathcal{E}(\varepsilon'),
\]

(S8)
where
\[ \mathcal{E}(\varepsilon) = \sqrt{\varepsilon^2 + \Delta^2(\varepsilon)}. \] (S9)

For the zero Fourier component we get
\[ (F_{\text{dis}}(\varepsilon)F_{\text{dis}}(\varepsilon'))_{q=0} = \frac{F_0(\varepsilon)F_0(\varepsilon')}{4\pi^3\nu^2 D} \frac{1}{\mathcal{E}(\varepsilon) + \mathcal{E}(\varepsilon')} \] . (S10)

The corresponding contribution to Eq. (S6) has the form
\[ \Phi^{(FF)} = \frac{L_0^2}{\Delta^4(T)} \frac{(2\pi T)^4}{4\pi^3\nu^2 D} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4 > 0} \lambda(\varepsilon_1,\varepsilon_2)\lambda(\varepsilon_3,\varepsilon_4) \frac{F_0(\varepsilon_1)F_0(\varepsilon_2)F_0(\varepsilon_3)F_0(\varepsilon_4)}{\mathcal{E}(\varepsilon_2) + \mathcal{E}(\varepsilon_4)}. \] (S11)

This expression is typical to fluctuation contributions to \( \Phi \). Among four energy summations, two are logarithmic involving large energies, \( \varepsilon \gg T_c \), while the other two come from \( \varepsilon \sim T_c \), where \( \Delta(\varepsilon) \) can be approximated by \( \Delta(T) \).

The latter summations introduce the dimensionless function
\[ N\left(\frac{T}{T_c}\right) = \frac{\Delta(T)}{\mathcal{E}_1\mathcal{E}_2(\mathcal{E}_1 + \mathcal{E}_2)} = \frac{2}{\pi} \int_0^\infty d\theta \frac{\tanh^2\left[\frac{\Delta(T)}{2T} \cosh \theta\right]}{\frac{\Delta(T)}{2\pi^3} \frac{1}{T}}, \quad T_c - T \ll T_c; \]
\[ T \ll T_c. \] (S12)

Performing summations over \( \varepsilon_2 \) and \( \varepsilon_4 \) in Eq. (S11) we get
\[ \Phi^{(FF)} = \frac{1}{16\pi^2 D \Delta^4(T)} L_0^2 \left(\frac{T}{T_c}\right) N \left(\frac{T}{T_c}\right) \left(2\pi T \sum_{\varepsilon > 0} \lambda(\varepsilon, T_c) F_0(\varepsilon)\right)^2. \] (S13)

Summation is done with the help of the SCE (4), and we obtain finally
\[ \Phi^{(FF)} = \frac{1}{16\pi^2 D \Delta^4(T)} L_0^2 \left(\frac{T}{T_c}\right) \left(2\pi T \sum_{\varepsilon > 0} \lambda(\varepsilon, T_c) F_0(\varepsilon)\right)^2. \] (S14)

**Correlation function \( \langle \delta \lambda_{\text{dis}} \delta \lambda_{\text{dis}} \rangle \)**

Mesoscopic fluctuations of the return probability which determines \( \delta \lambda(\varepsilon, \varepsilon') = -\lambda_0^2 \ln[1/\max(\varepsilon, \varepsilon')\tau] \) have been calculated for the normal state in Ref. [S2]:
\[ \langle \lambda_{\text{dis}}(\varepsilon_1, \varepsilon_2)\lambda_{\text{dis}}(\varepsilon_3, \varepsilon_4)\rangle \equiv \delta \lambda(\varepsilon_1, \varepsilon_2)\delta \lambda(\varepsilon_3, \varepsilon_4) \frac{1}{16\pi^3\nu^2 D} \frac{1}{|\varepsilon_{13}| + |\varepsilon_{14}| + |\varepsilon_{23}| + |\varepsilon_{24}|}, \] (S15)

where \( \varepsilon_{ij} \equiv \varepsilon_i + \varepsilon_j \).

Generalization to the superconducting case is achieved by replacing \( \varepsilon \rightarrow \mathcal{E}(\varepsilon) \). All four terms in Eq. (S15) equally contribute to \( \Phi \), and we get
\[ \Phi^{(\lambda\lambda)} = \frac{L_0^2}{\Delta^4(T)} \frac{(2\pi T)^4}{4\pi^3\nu^2 D} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4 > 0} \delta \lambda(\varepsilon_1,\varepsilon_2)\delta \lambda(\varepsilon_3,\varepsilon_4) \frac{F_0(\varepsilon_1)F_0(\varepsilon_2)F_0(\varepsilon_3)F_0(\varepsilon_4)}{\mathcal{E}(\varepsilon_2) + \mathcal{E}(\varepsilon_4)}. \] (S16)

Analogously to Eq. (S11)], summations over \( \varepsilon_2 \) and \( \varepsilon_4 \) yields \( N(T/T_c) \), while summations over \( \varepsilon_1 \) and \( \varepsilon_3 \) are converted to a logarithmic integral:
\[ \Phi^{(\lambda\lambda)} = \frac{1}{16\pi^2 D \Delta^4(T)} L_0^2 \left(\frac{T}{T_c}\right) N \left(\frac{T}{T_c}\right) \left(\int_0^{\varepsilon_c} \delta \lambda(\varepsilon, \varepsilon') \cosh \lambda_0(\varepsilon \varepsilon' - \varepsilon) d\varepsilon'\right)^2, \] (S17)
and hence
\[ \Phi^{(\lambda\lambda)} = \frac{1}{16\pi^2 D \Delta^4(T)} L_0^2 \left(\frac{T}{T_c}\right) N \left(\frac{T}{T_c}\right) [\cosh \lambda_0(\varepsilon_c - 1)^2]. \] (S18)
Correlation function $\langle F_{\text{dis}} \delta_{\text{dis}} \rangle$

The cross term is evaluated analogously:

$$\Phi^{(F\lambda)} = -2 \frac{1}{16\pi^2 D \Delta(T)} L_0^2 \left( \frac{T}{T_c} \right) N \left( \frac{T}{T_c} \right) [\cosh \lambda_y \zeta_* - 1].$$  \hfill (S19)

Resulting expression for $\langle \Delta_1 \Delta_1 \rangle$

Adding (S14), (S18) and (S19), we obtain

$$\Phi = -2 \frac{1}{16\pi^2 D \Delta(T)} K \left( \frac{T}{T_c} \right) \cosh^2 \lambda_y \zeta_* = -\frac{\pi D}{g(g - g_c) \Delta(T)} K \left( \frac{T}{T_c} \right),$$  \hfill (S20)

where we have introduced $K(T/T_c) = L_0^2(T/T_c) N(T/T_c)$, used the relation $g = 4\pi \nu D$, and employed $\cosh^2 \lambda_y \zeta_* = g/(g - g_c)$. With the help of Eq. (S6), one readily obtains Eq. (10) of the main text.

It’s worth noting that Eq. (S20) agrees with our previous results [S2]. Indeed, in the limit $T \to T_c$, Eq. (S20) can be simplified with the help of Eqs. (S4) and (S12) as

$$\langle \langle \Delta(T) \Delta(T) \rangle \rangle_{q=0} = \frac{2T^3}{7(3)\nu^2 D \Delta^2(T)} \cosh^2 \lambda_y \zeta_*.$$  \hfill (S21)

On the other hand, the same correlator can be obtained from the correlation function of the coefficient $\alpha$ in the Ginzburg-Landau (GL) expansion [S2]:

$$\langle (\alpha \alpha) \rangle_{q=0} = \frac{7(3)}{8\pi^4 DT} \cosh^2 \lambda_y \zeta_*$$  \hfill (S22)

with the help of the relation

$$\langle \langle \Delta(T) \Delta(T) \rangle \rangle_{q=0} = \frac{\langle (\alpha \alpha) \rangle_{q=0}}{4\beta^2 \Delta^2(T)},$$  \hfill (S23)

where $\beta = 7(3)\nu/(8\pi^2 T^2)$ is the nonlinear coefficient in the GL functional. One can easily verify that Eq. (S23) coincides with Eq. (S21).

III. DENSITY OF THE SUBGAP STATES

Introducing $\theta = \pi/2 + i\psi$, we rewrite Eq. (12) as

$$-\xi^2 \nabla^2 \psi + F(\psi) = -\frac{\delta \Delta(r) \sinh \psi}{\Delta_0},$$  \hfill (S24)

where $\xi^2 = D/2\Delta_0$, and

$$F(\psi) = -\frac{E}{\Delta_0} \cosh \psi + \sinh \psi - \eta \sinh \psi \cosh \psi.$$  \hfill (S25)

At the minigap ($E = E_g$), $\cosh \psi_g = \eta^{-1/3}$. For small deviation from the gap, the function $F(\psi)$ in the vicinity of its maximum can be written as

$$F(\psi) \approx \Omega(\psi - \psi_0) - \rho(\psi - \psi_0)^2,$$  \hfill (S26)

where

$$\Omega = \sqrt{6} \sqrt{1 - \eta^{2/3}} \sqrt{\frac{E_g - E}{\Delta_0}}, \quad \rho = \frac{3}{2} \eta^{1/3} \sqrt{1 - \eta^{2/3}}.$$  \hfill (S27)
Comparison of the linear in $\psi$ terms in the left-hand side of Eq. (S24) defines the relevant length scale

$$L_E = \frac{\xi}{\sqrt{\Omega}}. \quad (S28)$$

At the mean-field level, $\psi$ is real below the gap. Finite DOS corresponds to appearance of a nonzero $\text{Im} \psi$ due to a large negative fluctuation of $\delta \Delta(r)$. Its probability is given by

$$P \propto \exp \left(-\frac{1}{2f(0)} \int \delta \Delta^2(r) \, d^d r\right) \equiv e^{-S_d}, \quad (S29)$$

where we have used that the instanton size, $L_E$, exceeds the correlation length, $\xi_0$, of the order parameter fluctuations.

At the quantitative level, the instanton action $S$ can be estimated as follows. To produce a nonzero DOS at $E < E_g$, the optimal fluctuation of $\delta \Delta(r)$ should have the magnitude of $-(E_g - E)$ and the spacial extent of $L_E$, which immediately gives the estimate [S3]

$$S_d \sim \frac{\Delta_0^2 \xi^d}{f(0)} \left(\frac{E_g - E}{E_g}\right)^{2-d/4} \quad (S30)$$

To find the numerical coefficient in Eq. (S30), one has to solve the instanton equation. Measuring coordinates in terms of $L_E$ introduced in Eq. (S28), appropriately rescaling $\psi - \psi_0$, and replacing $\sinh \psi$ in the right-hand side of Eq. (S24) by its value at $E_g$, we rewrite Eq. (S24) in a universal dimensionless form:

$$-\nabla^2 \phi + \phi - \phi^2 = h(r), \quad (S31)$$

where

$$\delta \Delta(r) = -\frac{\Delta_0 \Omega^2}{\rho \sinh \psi_g} h(r/L_E). \quad (S32)$$

Minimization of the functional $\int h^2(r) \, d^d r$ leads to the fourth order differential equation for $\phi(r)$:

$$(-\nabla^2 + 1 - 2\phi)(-\nabla^2 \phi + \phi - \phi^2) = 0. \quad (S33)$$

The spherically symmetric optimal fluctuation solving (S33) in $d$ dimensions satisfies the second-order differential equation [S4]

$$-\nabla^2_{d-2} \phi + \phi - \phi^2 = 0, \quad (S34)$$

where $\nabla^2_{d-2} \equiv \partial^2 / \partial r^2 - (d-3)r^{-1} \partial / \partial r$ is the radial part of the Laplace operator in the $(d-2)$-dimensional space. The instanton is characterized by the number

$$a_d = \int h^2 d^d r = 4 \int \frac{\phi'^2}{r^2} \, d^d r = \begin{cases} 48\pi/5, & d = 3, \\ 4, & d = 2, \end{cases} \quad (S35)$$

where $a_d$ follows from the exact solution $\phi_3(r) = (3/2) \cosh^{-2}(r/2)$ [S3], while $a_2$ is obtained by a numerical solution of Eq. (S34).

Returning to the dimensional variables, we get for the instanton action in the limit $\eta \ll 1$:

$$S_d = \frac{8a_d \Delta_0^2 \xi^d}{6^{d/4} f(0)} \left(\frac{E_g - E}{E_g}\right)^{2-d/4} \quad (S36)$$

In the 2D case,

$$S_2 = \frac{4a_2 D \Delta_0}{\sqrt{6} f(0)} \left(\frac{E_g - E}{E_g}\right)^{3/2} \quad (S37)$$

leading to Eqs. (18) and (19).
IV. ROLE OF A FINITE FILM THICKNESS

In films with a finite thickness \(d \lesssim \xi_0\), there exists a contribution to the depairing parameter \(\eta\) in Eq. (13) coming from large wave vectors \((qd \gg 1)\), where diffusion is three-dimensional:

\[
\eta_{3D} = \frac{2}{\Delta} \int f(q) \frac{d^3q}{(2\pi)^3} = \frac{1}{\pi^2\Delta D} \int_1^1 f(q) dq
\]  
(S38)

[here \(\Delta \equiv \Delta(T)\) is the temperature-dependent BCS order parameter]. In this region, Coulomb effects are weak and all complications related with the energy dependence of \(\lambda\) and \(\Delta\) can be neglected. Equation (8) is then replaced by a simpler expression written for an arbitrary 3D wave vector:

\[
\delta\Delta(T, q) = L_q\left(\frac{T}{T_c}\right) 2\pi T \sum_{\epsilon > 0} F_{\text{dis}}(\epsilon, q),
\]  
(S39)

where \(L_q(T/T_c)\) is the BCS fluctuation propagator at finite momentum:

\[
L_q^{-1}(T/T_c) = 2\pi T \sum_{\epsilon > 0} \frac{\mathcal{E}(\epsilon) Dq^2 + 2\Delta^2(T) \mathcal{E}(\epsilon)}{\mathcal{E}^2(\epsilon) [Dq^2 + 2\mathcal{E}(\epsilon)]}.
\]  
(S40)

In the limit \(q\xi_0 \gg 1\), one recovers the known inverse logarithmic decay of the fluctuation propagator [S3]:

\[
L_q(T/T_c) \approx \frac{1}{\ln(Dq^2/T_c)}, \quad q\xi_0 \gg 1.
\]  
(S41)

With the help of Eq. (S7) the correlation function \(f(q) = \langle \delta\Delta\delta\Delta \rangle_q\) can be written as

\[
f(q) = L_q^2\left(\frac{T}{T_c}\right) \left(\frac{T\Delta}{\nu_3 D}\right)^2 \sum_{\epsilon_1, \epsilon_2 > 0} \int d^3k \frac{D_{\epsilon\epsilon'}(k + q/2)D_{\epsilon\epsilon'}(k - q/2)}{\mathcal{E}(\epsilon_1)\mathcal{E}(\epsilon_2)},
\]  
(S42)

where \(\nu_3 = \nu/d\) is the 3D DOS at the Fermi level. Integrating over \(k\) with the help of the Feynman’s trick we get

\[
f(q) = \frac{1}{8\pi^3} L_q^2\left(\frac{T}{T_c}\right) \left(\frac{T\Delta}{\nu_3 D}\right)^2 \sum_{\epsilon_1, \epsilon_2 > 0} \frac{1}{\mathcal{E}(\epsilon_1)\mathcal{E}(\epsilon_2)} \int_0^1 \frac{dx}{x(1-x)q^2 + (\mathcal{E}(\epsilon_1) + \mathcal{E}(\epsilon_2))/D} \approx \frac{1}{8\pi^2q} \left(\frac{\Delta}{\nu_3 D}\right)^2.
\]  
(S43)

In the limit \(q\xi_0 \gg 1\), only \(\epsilon_{1, 2} \gg T_c\) with \(\mathcal{E}(\epsilon) \approx |\epsilon|\) are important. Replacing summations by integrations and using Eq. (S41) we find

\[
f(q) = \frac{1}{8\pi^3} L_q^2\left(\frac{T}{T_c}\right) \left(\frac{\Delta}{\nu_3 D}\right)^2 \int_0^{Dq} \frac{d\xi_1}{\xi_1} \int_0^{Dq} \frac{d\xi_2}{\xi_2} \int_0^1 \frac{dx}{x(1-x)q^2 + (\epsilon_1 + \epsilon_2)/D} \approx \frac{1}{8\pi^2q} \left(\frac{\Delta}{\nu_3 D}\right)^2.
\]  
(S44)

The obtained \(1/q\) dependence of the correlation function \(f(q)\) leads in Eq. (S45) to the logarithmic contribution from the region \(l \ll r \ll d:\)

\[
\eta_{3D} = \frac{\Delta(T)}{8\pi^4 D} \left(\frac{1}{\nu_3 D}\right)^2 \ln \frac{d}{l},
\]  
(S45)

leading to Eq. (17).

For even thicker films with \(d \gg \xi_0\), the 2D contribution (15) is small and Eq. (S45) with \(d\) replaced by \(\xi_0\) gives the leading contribution to the depairing rate.