Quantum decay of the supercurrent and intrinsic capacitance of Josephson junctions beyond the tunnel limit

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A nondissipative supercurrent state of a Josephson junction is metastable with respect to the formation of a finite-resistance state. This transition is driven by fluctuations, thermal at high temperatures and quantum at low temperatures. We evaluate the lifetime of such a state due to quantum fluctuations in the limit when the supercurrent is approaching the critical current. The decay probability is determined by the instanton action for the superconducting phase difference across the junction. At low temperatures, the dynamics of the phase is massive and is determined by the effective capacitance, which is a sum of the geometric and intrinsic capacitance of the junction. We model the central part of the Josephson junction either by an arbitrary short mesoscopic conductor described by the set of its transmission coefficients, or by a diffusive wire of an arbitrary length. The intrinsic capacitance can generally be estimated as $C_* \sim G/E_g$, where G is the normal-state conductance of the junction and E_g is the proximity minigap in its normal part. The obtained capacitance is sufficiently large to qualitatively explain the hysteretic behavior of the current-voltage characteristic even in the absence of overheating.

I. INTRODUCTION

Macroscopic quantum tunneling is a fascinating manifestation of quantum-mechanical behavior in large-scale systems with many degrees of freedom. It is responsible for a finite lifetime of a metastable state that cannot decay classically by thermal activation at zero temperature. Since the 1980s, macroscopic quantum tunneling has been studied in a number of condensed-matter systems: various types of Josephson junctions [1–11], Josephson junction arrays [12], phase-slip centers [13,14], vortices in superconductors [15], small ferromagnetic particles [16,17], etc.

The theoretical description of macroscopic quantum phenomena is based on the concept of a collective coordinate, χ. In the simplest cases, it can be considered as a slow variable, which allows to integrate out the other electronic degrees of freedom and end up with an effective quantum mechanics for $\chi(t)$. The resulting dynamics of the collective degree of freedom is generically nonlocal in time since interaction with other modes produces the retardation effect, intimately related with dissipation. Intensive studies of dissipative quantum mechanics [18–20] have been triggered by the pioneering work of Caldeira and Leggett [1].

Among various systems, the Josephson junction can be considered as a prototypical model of macroscopic quantum tunneling. Here the superconducting phase difference across the junction, χ, plays the role of a collective coordinate. A nondissipative Josephson current can run through the system, described by a certain current-phase relation $I(\chi)$ [21], nonsinusoidal in general [Fig. 1(a)], with the critical (maximal) current I_c reached at some phase difference χ_c. The current can be obtained by differentiating the free energy of the junction: $I(\chi) = (2e/\hbar)\,\partial F_0(\chi)/\partial\chi$. In the current-biased regime with the driving current I, the equilibrium states ($\chi = \chi_1 \bmod 2\pi$)

correspond to the minima of the Legendre-transformed free energy, $F(\chi) = F_0(\chi) - (\hbar/2e)I\chi$, which has the standard form of a washboard potential shown in Fig. 1(b). The supercurrent state with $\chi = \chi_1$ is metastable and does decay (due to quantum or thermal fluctuations) into a resistive branch. Then the junction usually stays in the dissipative regime unless I is decreased to a smaller retrapping current, resulting in a hysteretic current-voltage characteristic [22].

Early studies of the supercurrent decay in Josephson junctions [2,3,20] assumed the tunnel limit, when the superconducting terminals are coupled through an insulating layer without its own electron dynamics (SIS junctions). Possible dissipation and charging effects were included phenomenologically by adding an Ohmic resistor and a capacitor in parallel with the junction [the resistively shunted junction (RSJ) model] [23]. Due to advances in nanofabrication technology, current experimental interest has turned toward the study of SNS junctions when two superconducting terminals are connected via a normal region [5,24–29] (including graphene [30,31] and the surface of a topological insulator [32–37]). Important new physics in this case is related to the superconducting proximity effect [38], which renders the normal part of the junction "partially superconducting." The strength of the proximity effect is characterized by the value of the spectral minigap (at $\chi = 0$), which can be estimated as $E_g \sim \min(\Delta, \hbar/\tau_{\mathrm{esc}})$, where Δ is the superconducting gap in the terminals and τ_{esc} is the time required for an electron in the normal region to establish a contact with superconductors [21,39]. For a diffusive wire of length L with good contacts to superconductors, $\hbar/\tau_{\mathrm{esc}}$ is of the order of the Thouless energy, $E_{\mathrm{Th}} = \hbar D/L^2$, where D is the diffusion coefficient.

Depending on the relation between Δ and $\hbar/\tau_{\mathrm{esc}}$, one can distinguish between short ($\hbar/\tau_{\mathrm{esc}} \gg \Delta$, with $E_g \approx \Delta$) and long ($\hbar/\tau_{\mathrm{esc}} \ll \Delta$, with $E_g \ll \Delta$) Josephson junctions. For short junctions, $I(\chi)$ can be expressed in terms of the transmission coefficients of the normal region [40], whereas long junctions require special treatment [41]. In both cases,
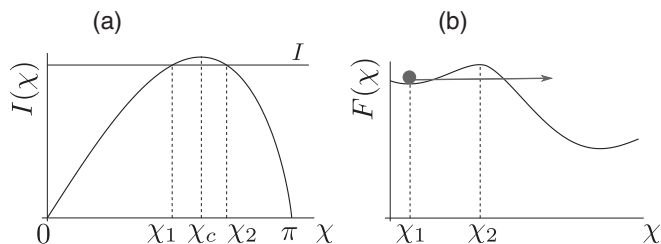
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FIG. 1. (a) Typical current-phase relation for a Josephson junction. (b) Free energy $F(\chi)$ of a current-biased junction vs the phase difference, χ, for the current I slightly below I_c.

the critical current can be written as

$$I_c = i_c G E_g / e, \qquad (1)$$

where G is the normal-state conductance of the junction, and $i_c \sim 1$ is a model-dependent factor. Equation (1) is a generalization of the Ambegaokar-Baratoff relation to SNS junctions.

In this paper, we take the first step toward the theory of supercurrent decay in Josephson junctions beyond the tunnel limit, and we calculate the lifetime of the supercurrent state due to *quantum* fluctuations. The normal region of the SNS junction will be modeled either by a short mesoscopic conductor characterized by an arbitrary set of transmission coefficients $\{T_\alpha\}$ or by a quasi-one-dimensional diffusive wire of an arbitrary length L (see Fig. 2). The spectral gaps in superconductors are assumed to be equal. We will work in the limit $I \rightarrow I_c$ (but not too close to I_c so that the WKB approximation is still applicable). In this limit, free energy $F(\chi)$ is flattened and χ becomes the slowest variable in the system, which makes it possible to treat electronic degrees of freedom in the adiabatic approximation (see the justification in Sec. IV B).

At low temperatures, $T \ll E_g$, thermal quasiparticles responsible for dissipation are frozen out and the junction can be described by the imaginary-time effective action with the capacitive dynamic term:

$$S[\chi(\tau)] = \int d\tau \left[\frac{C(\chi_c)}{2(2e)^2} \left(\frac{\partial \chi}{\partial \tau} \right)^2 + F(\chi) \right]. \qquad (2)$$

Such a local-in-time description of the phase dynamics is valid only in the supercurrent state with $I \rightarrow I_c$. The effective capacitance,

$$C(\chi) = C_{\text{geom}} + C_*(\chi), \qquad (3)$$

is a sum of the geometric, C_{geom}, and intrinsic, $C_*(\chi)$, capacitances of the junction. The latter is determined by the response
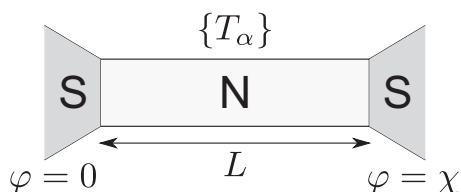

FIG. 2. SNS junction through a normal conductor. We consider two models of the normal part: (i) an arbitrary short scatterer with the transmission coefficients $\{T_\alpha\}$ (Sec. II), and (ii) a quasi-one-dimensional diffusive wire of an arbitrary length L (Sec. III).

TABLE I. Various properties of some model Josephson junctions ($E_g = \Delta$ marks short junctions), and the resulting values for the capacitance coefficient, $\alpha_C(\chi_c) = C_*(\chi_c)/(G/E_g)$.

	Tunnel	Short wire	Chaotic QD	Long wire
E_g	Δ	Δ	Δ	$3.12\, E_{\text{Th}}$
$\chi_c/(\pi/2)$	1	1.255	1.299	1.27
i_c	$\pi/2$	2.082	2.186	3.47
i_c''	$\pi/2$	1.706	1.791	2.92
$\alpha_C(\chi_c)$	$3\pi/32$	0.589	0.689	0.905

of the Andreev bound states to nonstationary boundary conditions. These states are known to be responsible for carrying the supercurrent [42]. At lowest temperatures, their low-frequency dynamics cannot be damped (with the kernel $|\omega|$ in terms of Matsubara frequency) as it is not related to any dissipation, and thus it should be capacitive (with the kernel ω^2).

Thus the problem of quantum decay of a nearly critical supercurrent reduces to calculating the intrinsic capacitance C_* of the junction (evaluated at the critical phase $\chi = \chi_c$). The intrinsic capacitance of the tunnel (SIS) junction was obtained in the works of Larkin and Ovchinnikov [2] and Ambegaokar, Eckern, and Schön [43]:

$$C_*^{\text{tun}}(\chi) = (3\pi G/32\Delta)(1 - \cos \chi) \qquad (4)$$

(here G is the tunnel conductance of the barrier). This result has been recently extended to the case of arbitrary short junctions by Galaktionov and Zaikin [44]. They derived the general expression for $C_*^{\text{short}}(\chi)$ in terms of transmission coefficients T_α [see Eq. (11)] and found that, quite generally, $C_*^{\text{short}}(\chi_c) \sim G/\Delta$. We will generalize that result further and show that the magnitude of $C_*(\chi)$ for an arbitrary SNS junction is determined by the ratio of the normal-state conductance G to the minigap E_g induced in the normal part due to the proximity effect [cf. Eq. (1) for I_c]:

$$C_*(\chi) = \alpha_C(\chi) G / E_g. \qquad (5)$$

The dimensionless capacitance coefficient $\alpha_C(\chi)$ is model-dependent. At the critical phase difference, $\alpha_C(\chi_c)$ is of the order of 1, and we will evaluate it for an arbitrary short scatterer using Galaktionov-Zaikin theory (see Table I) and for a quasi-one-dimensional diffusive wire with the help of the nonlinear σ model (see Figs. 3 and 4).

In the case when the geometric capacitance can be neglected ($C_{\text{geom}} \ll C_*$), we obtain that the instanton action, which determines the lifetime $\tau_0 \propto \exp(S)$ of the supercurrent state, depends only on the dimensionless conductance of the normal part,

$$S = \gamma \frac{G}{G_Q} \left(1 - \frac{I}{I_c} \right)^{5/4}, \qquad (6)$$

where $G_Q = e^2/\pi\hbar$ is the conductance quantum, and the coefficient $\gamma \sim 1$ is model-dependent. Its values for several model Josephson junctions are given by

$$\gamma = \begin{cases} 0.874 & \text{(tunnel barrier)}, \\ 1.65 & \text{(short diffusive wire)}, \\ 2.59 & \text{(long diffusive wire)}, \\ 1.83 & \text{(ballistic chaotic quantum dot)}. \end{cases} \qquad (7)$$
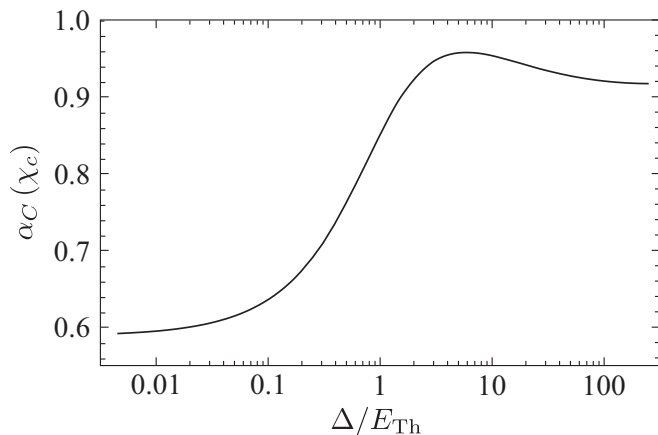
FIG. 3. The capacitance coefficient at the critical phase, $\alpha_C(\chi_c) = C_*(\chi_c)/(G/E_g)$, of the Josephson junction made of a finite-size wire vs Δ/E_{Th}. It is changing from $\alpha_C^{\text{short}}(\chi_c) = 0.589$ for short wires to $\alpha_C^{\text{long}}(\chi_c) = 0.905$ for long wires.

The paper is organized as follows. The zero-temperature intrinsic capacitance of a short SNS junction with an arbitrary normal region specified by its transmission coefficients is discussed in Sec. II on the basis of the Galaktionov-Zaikin formula. In Sec. III we present an approach based on the nonlinear σ model that allows us to determine the intrinsic capacitance of the Josephson junction through a diffusive quasi-one-dimensional normal wire of an arbitrary length, tracing its behavior from short- to long-wire limits. The resulting expression for the lifetime of a slightly subcritical current due to quantum fluctuations is presented in Sec. IV, where we also discuss the limits of applicability of our approach. In the concluding Sec. V, we summarize our results and discuss them in the context of the experimentally observed hysteresis in extended Josephson junctions. Finally, numerous technical details are relegated to several Appendixes. In our systems of units, $\hbar = k_B = 1$.
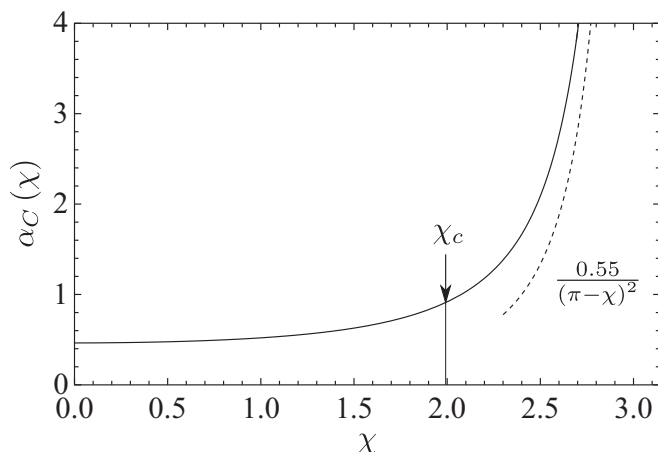

FIG. 4. The capacitance coefficient $\alpha_C(\chi) = C_*(\chi)/(G/E_g)$ of the long ($E_{\text{Th}} \ll \Delta$) Josephson junction through a quantum wire vs the phase difference across the junction, χ. The critical phase, $\chi_c = 1.27(\pi/2)$, is marked by the arrow. The dashed line shows the asymptotic behavior (41) near $\chi = \pi$.

II. SHORT MESOSCOPIC CONDUCTOR

A. Scattering matrix approach

In this section, we consider the case when the normal part can be described by the set of its transmission eigenvalues T_α with the distribution function

$$\mathcal{P}(T) = \left\langle \sum_\alpha \delta(T - T_\alpha) \right\rangle. \tag{8}$$

The scattering matrix theory proved to be a powerful and intuitively clear method for studying quantum transport in mesoscopic conductors [45]. For normal systems, its most renowned predictions are the celebrated Landauer formula for the dc conductance [46], $G = G_Q \sum_\alpha T_\alpha$, and the expression for the Fano factor in the theory of shot noise [47], $F = \sum_\alpha T_\alpha(1 - T_\alpha)/\sum_\alpha T_\alpha$.

Application of the scattering matrix approach to the superconducting hybrid system is a more delicate issue [40,45]. It requires the elements of the normal-state scattering matrix to be energy-independent on the relevant energy scale set by the minigap E_g [41]. Since the scattering matrix acquires energy dependence at the scale $\hbar/\tau_{\text{esc}}$, such a situation is realized for sufficiently short junctions with $\hbar/\tau_{\text{esc}} \gg \Delta$ and hence $E_g \approx \Delta$ (a diffusive SNS junction with ideal interfaces belongs to this class if $E_{\text{Th}} \gg \Delta$). Then the Josephson current can be found with the help of the Beenakker formula [40]:

$$I(\chi) = \frac{e\Delta^2}{2} \sin\chi \sum_\alpha \frac{T_\alpha}{\epsilon_\alpha} \tanh\frac{\epsilon_\alpha}{2T}, \tag{9}$$

where T is the temperature and ϵ_α is the energy of the Andreev bound state in the corresponding channel:

$$\epsilon_\alpha = \Delta\sqrt{1 - T_\alpha \sin^2(\chi/2)}. \tag{10}$$

The phase dynamics of a short Josephson junction was studied recently by Galaktionov and Zaikin [44]. With the help of the Keldysh technique, they derived an effective action for small phase fluctuations near the equilibrium phase χ. Slow phase dynamics in the zero-temperature limit is governed by the capacitance (3), where the intrinsic capacitance is given by

$$\begin{aligned} C_*(\chi) = \frac{e^2}{4\Delta} \sum_\alpha \Bigg\{ & \frac{2 - (2 - T_\alpha)\sin^2(\chi/2)}{T_\alpha \sin^4(\chi/2)} \\ & - [1 - T_\alpha \sin^2(\chi/2)]^{-5/2}\Bigg[2T_\alpha(T_\alpha - 2)\sin^2(\chi/2) \\ & + 5 + T_\alpha + \frac{2 - 2(1 + 2T_\alpha)\sin^2(\chi/2)}{T_\alpha \sin^4(\chi/2)} \Bigg] \Bigg\}. \end{aligned} \tag{11}$$

Since the resulting expression is sufficiently involved, we find it instructive to rederive Eq. (11) in the Matsubara representation. Though the general line of the derivation is very similar to that of Ref. [44], the absence of an additional Keldysh matrix structure makes it possible to track the details of the calculation. This procedure summarized in Appendix A reproduces Eq. (11) obtained by Galaktionov and Zaikin.

B. Intrinsic capacitance for model junctions

Equation (11) can be used to evaluate numerically the intrinsic capacitance, $C_*(\chi)$, of various short Josephson

junctions [44]. We calculate it at the critical phase, $C_*(\chi_c)$, for three types of structures, with the superconducting terminals coupled through the following links:

(i) A tunnel barrier with all $T_\alpha \ll 1$.

(ii) A short ($E_{\text{Th}} \gg \Delta$) diffusive wire, with $\mathcal{P}(T)$ given by Dorokhov distribution [48],

$$\mathcal{P}(T) = \frac{G}{2G_Q} \frac{1}{T\sqrt{1-T}}. \qquad (12)$$

(iii) A ballistic chaotic quantum dot [49], with

$$\mathcal{P}(T) = \frac{2G}{\pi G_Q} \frac{1}{\sqrt{T(1-T)}}. \qquad (13)$$

The results along with some characteristics of these Josephson junctions are presented in Table I. We see that the capacitance coefficient $\alpha_C(\chi_c)$ defined in Eq. (5) is generally of the order of unity, depending on the particular distribution of transmission coefficients $\mathcal{P}(T)$.

III. QUASI-ONE-DIMENSIONAL WIRE

Here we calculate the intrinsic capacitance of the SNS junction made of a normal diffusive wire coupled to superconductors through highly transparent interfaces. The scattering-matrix approach used in the previous section cannot be applied to sufficiently long wires ($E_{\text{Th}} \lesssim \Delta$). In this limit, the energy dispersion of the scattering matrix at relevant energies $E \sim E_{\text{Th}}$ is not negligible, and the action does not have a simple form of Eq. (A1) [50]. To find $C_*(\chi)$, one then has to use a more general method of the nonlinear σ model and study spatially inhomogeneous configurations of the matrix field Q in the wire.

A. Diffusive σ model and Usadel equation

The superconducting proximity effect in a diffusive metal can be conveniently described by the replica σ model in imaginary time [51,52]. Its action for the quasi-one-dimensional metallic wire of Fig. 2 can be written as

$$S[Q] = \frac{G}{16G_Q} \int_{-1/2}^{1/2} dr \, \text{Tr}[(\nabla Q)^2 - 4\varepsilon\tau_3 Q], \qquad (14)$$

where the spacial coordinate r is measured in units of the wire length L, summation goes over Matsubara energies $\epsilon = \pi T(2n+1)$, and $\varepsilon = \epsilon/E_{\text{Th}}$ with the Thouless energy $E_{\text{Th}} = \hbar D/L^2$. In general, the field Q subject to the constraint $Q^2 = 1$ acts as a matrix in the replica space, but since the phase difference χ carries no replica index, the structure of the instanton describing quantum tunneling is trivial in the replica space, and we can omit the replica index everywhere. Then $Q = Q_{\epsilon_1,\epsilon_2}(r)$ becomes a matrix in the Nambu-Gor'kov space (Pauli matrices τ_i), as in Appendix A.

The stationary saddle point $Q_0(\epsilon_1,\epsilon_2;r) = 2\pi\delta(\epsilon_1 - \epsilon_2)Q_0(\epsilon_1;r)$ for the σ-model action (14) satisfies the Usadel equation [53],

$$-(Q_0 Q_0')' + \varepsilon[\tau_3, Q_0] = 0, \qquad (15)$$

where a prime denotes the derivative with respect to r. Assuming perfect NS interfaces, we write the boundary conditions with the antisymmetric choice of superconducting

phases as $Q_0(\epsilon; \pm 1/2) = Q_S^{(\pm\chi/2)}(\epsilon)$, where $Q_S^{(\varphi)}(\epsilon)$ is defined in Eq. (A3).

In the standard parametrization in terms of the spectral angles $\theta_\epsilon(r)$ and $\varphi_\epsilon(r)$,

$$Q_0 = (\tau_1 \cos\varphi - \tau_2 \sin\varphi)\sin\theta + \tau_3 \cos\theta, \qquad (16)$$

the Usadel equation reduces to two coupled equations:

$$(\sin^2\theta \, \varphi')' = 0, \qquad (17a)$$

$$\theta'' - 2\varepsilon\sin\theta - (\varphi')^2\sin\theta\cos\theta = 0, \qquad (17b)$$

with the boundary conditions at $r = \pm 1/2$:

$$\theta(\pm 1/2) = \theta_S(\epsilon) = \arctan\Delta/\epsilon, \quad \varphi(\pm 1/2) = \pm\chi/2. \qquad (18)$$

The stationary supercurrent is given by

$$I(\chi) = \frac{\pi G}{e} T \sum_\epsilon \sin^2\theta_\epsilon(r) \, \varphi_\epsilon'(r) \qquad (19)$$

[its conservation is guaranteed by Eq. (17a)]. The temperature and length dependence of the critical current of diffusive SNS junctions were studied in Ref. [24]. Our further analysis will be limited to the $T = 0$ case.

B. Perturbative expansion near the saddle point

In the presence of a time-dependent phase difference across the junction, $\chi(\tau)$, the Q matrices in the leads become functions of two time arguments [cf. Eqs. (A4)]:

$$Q_{R(L)}(\tau_1,\tau_2) = e^{\pm i\chi(\tau_1)\tau_z/2} Q_S^{(0)}(\tau_1 - \tau_2) e^{\mp i\chi(\tau_2)\tau_z/2}. \qquad (20)$$

To get the action as a functional of $\chi(\tau)$, one has to integrate out $Q(r)$ in the wire with the boundary conditions (20). Near the criticality, $I \to I_c$, variations of $x(\tau) = \chi(\tau) - \chi_c$ on the instanton trajectory are small, and hence induced (nondiagonal in energy) deviations of Q from the saddle-point Q_0 can be treated perturbatively.

Perturbative expansion of the diffusive σ model near the supercurrent state in Josephson junctions has been developed in Ref. [54], and here we generalized it to the case of nonstationary boundary conditions (an alternative approach would be to work on the level of the dynamic Usadel equations, as was done in Ref. [55]). In that paper, the authors studied the real part of admittance for the case of the long diffusive SNS junction, while the intrinsic capacitance can be extracted from the low-frequency expansion of its imaginary part.

First we present the stationary saddle point $Q_0(r)$ in the form $Q_0 = U^{-1}\tau_1 U$ with the diagonal-in-energy position-dependent matrix $U_\epsilon(r) = e^{-i\tau_2[\pi/4 - \theta_\epsilon(r)/2]}e^{-i\tau_3\varphi_\epsilon(r)/2}$. Then we parametrize small fluctuations near Q_0 as

$$Q = U^{-1}\tau_1(1 + W + W^2/2 + \cdots)U \qquad (21)$$

in terms of the field W satisfying $\{\tau_1, W\} = 0$ and $W^\dagger = -W$. The former constraint implies $W = c\tau_2 + d\tau_3$, and we combine c and d into a vector object,

$$w_{\epsilon_1\epsilon_2}(r) = V_{\epsilon_1\epsilon_2}^{-1}(r)\begin{pmatrix} c_{\epsilon_1\epsilon_2}(r) \\ d_{\epsilon_1\epsilon_2}(r) \end{pmatrix}, \qquad (22)$$

where the unitary matrix V is introduced to simplify the quadratic action [Eq. (29) below] [54]. Its explicit form can be

found in Appendix B. The Pauli matrices acting in the (c,d) space will be referred to as Σ_i [56]. Physically, the field w describes soft diffusive modes on top of a superconducting state (in the presence of a supercurrent they can no longer be classified as diffusons and cooperons).

Expansion of the action in powers of w reads

$$S[Q] = S[Q_0] + S^{(w)}[w] + S^{(w^2)}[w] + \cdots, \qquad (23)$$

where each term is a combination of the bulk and boundary contributions:

$$S^{(w^n)}[w] = S_{\text{bulk}}^{(w^n)}[w] + S_{\text{bound}}^{(w^n)}[w]. \qquad (24)$$

Since Q_0 is the saddle point, the linear term $S^{(w)}$ is totally due to the boundary, but it will contribute to the quadratic-in-x action [see Eq. (36) below]. The boundary contributions are given by

$$S_{\text{bound}}^{(w)} = \frac{iG}{2G_Q} \int \frac{d\epsilon}{2\pi} b_\epsilon^T w_{\epsilon,\epsilon} \Big|_{-1/2}^{1/2}, \qquad (25)$$

$$S_{\text{bound}}^{(w^2)} = \frac{G}{4G_Q} \int \frac{d\epsilon_1}{2\pi} \frac{d\epsilon_2}{2\pi} w_{\epsilon_1,\epsilon_2}^\dagger B_{\epsilon_1,\epsilon_2} w_{\epsilon_1,\epsilon_2} \Big|_{-1/2}^{1/2}, \qquad (26)$$

where $w_{\epsilon_1,\epsilon_2}^\dagger \equiv (w_{\epsilon_1,\epsilon_2})^\dagger$, and the operators b and B read

$$b_\epsilon(r) = V_{\epsilon,\epsilon}^T \begin{pmatrix} -\theta_\epsilon' \\ \sin\theta_\epsilon\,\varphi_\epsilon' \end{pmatrix}, \qquad (27)$$

$$B_{\epsilon_1,\epsilon_2}(r) = \nabla + i\frac{\cos\theta_{\epsilon_1}\varphi_{\epsilon_1}' + \cos\theta_{\epsilon_2}\varphi_{\epsilon_2}'}{2}\Sigma_2. \qquad (28)$$

The bulk contribution is given by

$$S_{\text{bulk}}^{(w^2)} = \frac{G}{4G_Q} \int \frac{d\epsilon_1}{2\pi} \frac{d\epsilon_2}{2\pi} \int_{-1/2}^{1/2} dr\, w_{\epsilon_1\epsilon_2}^\dagger \big(-\nabla^2 + \Omega_{\epsilon_1\epsilon_2}\big) w_{\epsilon_1\epsilon_2}, \qquad (29)$$

where an explicit expression for the effective potential Ω can be found in Appendix B.

C. Effective capacitance evaluation

The effective action for the phase variable $x(\tau)$ is determined by the saddle-point trajectory of the action (23) with the boundary conditions (20). It can be written as a series in x:

$$w = w^{(1)} + w^{(2)} + \cdots, \qquad (30)$$

with $w^{(n)} \propto x^n$. As we are interested in quadratic-in-x terms, it suffices to follow only $w^{(1)}$ and $w^{(2)}$. Due to the nonlinearity of the theory, the linear terms $w^{(1)}$ may influence the equation of motion for the quadratic term $w^{(2)}$. However, as shown in Appendix C, this mechanism gives no contribution to the effective action. Therefore, we may consider linear equations of motion obtained by varying $S_{\text{bulk}}^{(w^2)}$:

$$[-\nabla^2 + \Omega_{\epsilon_1,\epsilon_2}(r)]w_{\epsilon_1,\epsilon_2}(r) = 0, \qquad (31)$$

and process each order in x independently.

The boundary conditions for Eq. (31) can be obtained by comparing Eq. (20) with the parametrization (21) at $r = \pm 1/2$.

The first two terms are given by

$$w_{\epsilon_1,\epsilon_2}^{(1)}(\pm 1/2) = \xi_{R(L)}^{(1)}(\epsilon_1,\epsilon_2)x_{\epsilon_1-\epsilon_2}, \qquad (32a)$$

$$w_{\epsilon_1,\epsilon_2}^{(2)}(\pm 1/2) = \int \frac{d\epsilon}{2\pi} \xi_{R(L)}^{(2)}(\epsilon_1,\epsilon_2,\epsilon)x_{\epsilon_1-\epsilon}x_{\epsilon-\epsilon_2}, \qquad (32b)$$

where

$$\xi_{R(L)}^{(1)}(\epsilon_1,\epsilon_2) = \mp\frac{i}{2}V_{\epsilon_1,\epsilon_2}^{-1}(\pm 1/2)\begin{pmatrix} 0 \\ \sin\vartheta_{\epsilon_1,\epsilon_2} \end{pmatrix}, \qquad (33a)$$

$$\xi_{R(L)}^{(2)}(\epsilon_1,\epsilon_2,\epsilon) = -\frac{i}{16}V_{\epsilon_1,\epsilon_2}^{-1}(\pm 1/2)\begin{pmatrix} \sin(\vartheta_{\epsilon_1,\epsilon} + \vartheta_{\epsilon_2,\epsilon}) \\ 0 \end{pmatrix}, \qquad (33b)$$

and $\vartheta_{\epsilon,\epsilon'} = [\theta_S(\epsilon) + \theta_S(\epsilon')]/2$. In what follows, we will need only diagonal elements of $w^{(2)}$, so we introduce the notation $\xi_{R(L)}^{(2)}(\epsilon_1,\epsilon_2) = \xi_{R(L)}^{(2)}(\epsilon_1,\epsilon_1,\epsilon_2)$.

With the boundary conditions (33), the solution of Eq. (31) can be written in the form

$$w_{\epsilon_1,\epsilon_2}^{(1)}(r) = \xi_{\epsilon_1,\epsilon_2}^{(1)}(r)x_{\epsilon_1-\epsilon_2}, \qquad (34a)$$

$$w_{\epsilon_1,\epsilon_1}^{(2)}(r) = \int \frac{d\epsilon_2}{2\pi} \xi_{\epsilon_1,\epsilon_2}^{(2)}(r)x_{\epsilon_1-\epsilon_2}x_{\epsilon_2-\epsilon_1}, \qquad (34b)$$

where the functions $\xi_{\epsilon_1,\epsilon_2}^{(1)}(r)$ and $\xi_{\epsilon_1,\epsilon_2}^{(2)}(r)$ obey the same Eq. (31) as $w_{\epsilon_1,\epsilon_2}(r)$ and satisfy the boundary conditions

$$\xi_{\epsilon_1,\epsilon_2}^{(1)}(\pm 1/2) = \xi_{R(L)}^{(1)}(\epsilon_1,\epsilon_2), \qquad (35a)$$

$$\xi_{\epsilon_1,\epsilon_2}^{(2)}(\pm 1/2) = \xi_{R(L)}^{(2)}(\epsilon_1,\epsilon_2). \qquad (35b)$$

To get the quadratic-in-x part of the action, we substitute Eqs. (34) into Eq. (23):

$$S^{(2)} = S^{(w^2)}[w^{(1)}] + S^{(w)}[w^{(2)}], \qquad (36)$$

which takes the form of Eq. (A5) with the kernel $\mathcal{K}(\epsilon,\omega)$ after the substitution $\epsilon = (\epsilon_1 + \epsilon_2)/2$ and $\omega = \epsilon_1 - \epsilon_2$. Then the intrinsic capacitance should be determined from Eq. (A6). To find it numerically, we expand the matrix $\Omega_{\epsilon+\omega/2,\epsilon-\omega/2}(r)$ and the solutions $\xi_{\epsilon+\omega/2,\epsilon-\omega/2}^{(n)}(r)$ in powers of ω as

$$\Omega_{\epsilon+\omega/2,\epsilon-\omega/2}(r) = \sum_{k=0}^\infty \Omega_k(\epsilon,r)\omega^k, \qquad (37)$$

$$\xi_{\epsilon+\omega/2,\epsilon-\omega/2}^{(n)}(r) = \sum_{k=0}^\infty \xi_k^{(n)}(\epsilon,r)\omega^k, \qquad (38)$$

and we write down the system of linear equations for the first three coefficients:

$$\begin{aligned} (-\nabla^2 + \Omega_0)\xi_0^{(n)} &= 0, \\ (-\nabla^2 + \Omega_0)\xi_1^{(n)} &= -\Omega_1\xi_0^{(n)}, \\ (-\nabla^2 + \Omega_0)\xi_2^{(n)} &= -\Omega_1\xi_1^{(n)} - \Omega_2\xi_0^{(n)}, \end{aligned} \qquad (39)$$

which should be supplemented by the boundary conditions obtained from the expansion of $\xi_{R(L)}^{(n)}(\epsilon + \omega/2, \epsilon - \omega/2)$ in ω.

Extracting $\partial^2\mathcal{K}(\epsilon,\omega)/\partial\omega^2$ from Eq. (36), we obtain

$$\frac{\partial^2\mathcal{K}(\epsilon,\omega)}{\partial\omega^2}\bigg|_{\omega=0} = 4ib^T\xi_2^{(2)}\bigg|_{-1/2}^{1/2} + 2\sum_{k=0}^{2}\xi_k^{(1)T}\nabla\xi_{2-k}^{(1)}\bigg|_{-1/2}^{1/2} \tag{40}$$

[the term originating from the Σ_2 part of Eq. (28) equals zero]. According to Eq. (A6), the intrinsic capacity of the junction is determined by the integral of Eq. (40) over ϵ.

D. Notes on numerical evaluation

The first step of the numerical simulation is to find the solutions for the Usadel equations (17). One way is to use the explicit expression in terms of elliptic functions [54], which reduces to finding two constants from a system of algebraic equations numerically. However, in order to obtain the functions $\theta_\epsilon(r)$ and $\varphi_\epsilon(r)$ numerically, we find it more convenient to follow the other way and to simulate the original differential Eqs. (17) directly. Obtained spectral angles are used to determine the matrices Ω_0, Ω_1, and Ω_2 in Eq. (37). Then the system (39) is solved numerically, the solutions are substituted into Eq. (40), and the integral over ϵ in Eq. (A6) is calculated with a proper energy grid.

As a check of the numerical method, it is instructive to calculate $\int d\epsilon\,\mathcal{K}(\epsilon,0)$. It is easy to see that it should be proportional to the derivative of the current-phase relation $\partial I/\partial\chi$. By checking that this quantity indeed crosses zero at the critical phase χ_c, we can provide an independent test for our numerical calculations.

E. Results

The resulting dependence of the capacitance coefficient $\alpha_C(\chi_c) = C_*(\chi_c)/(G/E_g)$ evaluated at the critical phase difference, χ_c, on the ratio $\Delta/E_{\text{Th}} \propto L^2$ is presented in Fig. 3 (for the dependence of E_g on Δ/E_{Th}, see, e.g., Fig. 5 of Ref. [54]). In general, $\alpha_C(\chi_c)$ is of the order of unity, varying from $\alpha_C^{\text{short}}(\chi_c) = 0.589$ for short wires to $\alpha_C^{\text{long}}(\chi_c) = 0.905$ for long wires [in that limit, $E_g = 3.12\,E_{\text{Th}}$ (Ref. [57]) and $I_c = 10.82\,GE_{\text{Th}}/e$ (Ref. [24])]. The maximal value of $\alpha_C^{\text{max}}(\chi_c) = 0.958$ is achieved at $\Delta/E_{\text{Th}} = 5.8$, corresponding to the wire length $L = 2.4\sqrt{D/\Delta}$.

For completeness, in Fig. 4 we present the phase dependence of the intrinsic capacitance coefficient $\alpha_C(\chi) = C_*(\chi)/(G/E_g)$ in the long-wire limit. In the vicinity of $\chi = \pi$, where the gap closes, the intrinsic capacitance diverges as

$$C_*^{\text{long}}(\chi) \approx \frac{0.55}{(\pi-\chi)^2}\frac{G_N}{E_g}. \tag{41}$$

A similar behavior obtained for short diffusive junctions in Ref. [44] is attributed to the presence of almost open channels ($T_\alpha \to 1$) in Dorokhov distribution (12).

IV. LIFETIME OF THE SUPERCURRENT STATE

A. Instanton action

Decay of the dissipationless supercurrent state governed by the action (2) is equivalent to a quantum-mechanical tunneling of a massive particle under a barrier. Near the criticality, at

$I \to I_c$, the potential barrier $\Delta F(x) = F(\chi_c + x) - F(\chi_c)$ can be approximated by a cubic parabola:

$$\Delta F(x) = \frac{G}{G_Q}\frac{E_g}{2\pi}\left[i_c\left(1-\frac{I}{I_c}\right)x - \frac{i_c''}{6}x^3\right], \tag{42}$$

where i_c is defined in Eq. (1), and $i_c'' = -I''(\chi_c)/(GE_g/e)$ is the dimensionless curvature of the current-phase characteristic at χ_c. Parameters i_c and i_c'' for some model Josephson junctions are listed in Table I. The instanton trajectory for the potential (42) takes the form

$$\chi(\tau) - \chi_1 = \frac{3\times(2i_ci_c'')^{1/2}(1-I/I_c)^{1/2}}{\cosh^2(\omega_p\tau/2)}, \tag{43}$$

where ω_p is the plasma frequency describing small phase oscillations near the minimum of $\Delta F(x)$. In the short-wire limit, ω_p was calculated in Ref. [44], and the general expression is given by

$$\omega_p = 2^{3/4}(i_ci_c'')^{1/4}\sqrt{\frac{E_gG}{C(\chi_c)}}\left(1-\frac{I}{I_c}\right)^{1/4}. \tag{44}$$

The lifetime of the subcritical current state in the WKB approximation can be estimated as

$$\tau_0 \approx (2\pi/\omega_p)\,e^S, \tag{45}$$

where S is the instanton action,

$$S = \frac{12\times2^{3/4}}{5\pi}\frac{i_c^{5/4}}{(i_c'')^{3/4}}\frac{\sqrt{C(\chi_c)E_gG}}{G_Q}\left(1-\frac{I}{I_c}\right)^{5/4}. \tag{46}$$

This expression generalizes the result for the tunnel limit [2] to the case of an arbitrary Josephson junction.

Equations (44)–(46) supplemented by the expressions (3) and (5) provide a full description of the supercurrent decay due to quantum fluctuations. The numerical coefficients i_c, i_c'', and $\alpha_C(\chi_c)$ for a number of model Josephson junctions are summarized in Table I.

If the geometric capacitance is sufficiently small, $C_{\text{geom}} \ll C_*(\chi_c)$, the instanton action (46) can be simplified. Substituting $C_*(\chi_c)$ from Eq. (5), we arrive at Eq. (6), with the model-dependent parameter

$$\gamma = \frac{12\times2^{3/4}}{5\pi}\frac{i_c^{5/4}\alpha_C^{1/2}(\chi_c)}{(i_c'')^{3/4}}. \tag{47}$$

For some model cases, the coefficient γ is given by Eq. (7).

B. Applicability of the theory

In our analysis, we rely on the adiabatic approximation for $\chi(\tau)$ justified at $I \to I_c$. In this limit, variation of the phase is weak and small, which allows us to expand the action. On the other hand, in the very vicinity of I_c, the instanton action $S \lesssim 1$ and the WKB method fails. Therefore, the present theory is applicable as long as $[G_Q^2/C(\chi_c)E_gG]^{2/5} \ll 1 - I/I_c \ll 1$. If the geometric capacitance is negligible, that reduces to

$$(G_Q/G)^{4/5} \ll 1 - I/I_c \ll 1. \tag{48}$$

Finally, we discuss the condition on the temperature range when the $T = 0$ description of the phase tunneling is applicable. The principal limitation is related to the

crossover to thermal decay with the activation exponent $e^{-\delta F/T}$, where $\delta F \sim (G/G_Q)E_g(1 - I/I_c)^{3/2}$ is the height of the free-energy barrier. Quantum description is applicable as long as $\delta F/T \gg S$, which translates into the constraint $[C(\chi_c)T^2/GE_g]^2 \ll 1 - I/I_c$. In the important limit of a small geometric capacitance, this condition reduces to a simple inequality

$$(T/E_g)^4 \ll 1 - I/I_c. \qquad (49)$$

The condition (49) also guarantees that the phase action can still be written in a capacitive form (2) neglecting quasiparticle damping effects. Indeed, dissipative terms in the action can be roughly described by a large shunting resistance $R(T) \sim G^{-1}e^{E_g/T}$, which has no effect on phase tunneling since the corresponding RC time is much larger than the period of plasma oscillations.

To estimate the predictions of our theory, consider a 1-μm-long SNS junction similar to that fabricated in Refs. [25,28,29]. With the parameters $G^{-1} \sim 5\,\Omega$, $E_{\mathrm{Th}} \sim 30\,\mathrm{mK}$, and $\Delta/E_{\mathrm{Th}} \sim 10^2$, the intrinsic capacitance is expected to be of the order of 10 pF. The condition (48) determines a wide range of allowed bias currents: $10^{-3} \ll 1 - I/I_c \ll 1$, in which the predicted lifetime (when the intrinsic capacitance dominates) varies from nanoseconds to practically unlimited values. For example, the decay time $\tau_0 \sim 1\,\mathrm{s}$ is achieved at $1 - I/I_c \sim 10^{-2}$. According to the constraint (49), the quantum tunneling regime is realized in the sub-100 mK temperature range.

Our calculation was performed in the assumption of ideally transparent SN interfaces, whereas experimentally studied junctions may have a finite conductance G_T of interfaces. Our theory is valid as long as $G_T \gg G$, but it can be straightforwardly extended beyond this limit by adding the corresponding boundary terms [52,58] to the σ-model action (14). In the opposite limit, $G_T \ll G$, the normal region can be treated as a chaotic quantum dot, as was done in Ref. [54].

V. DISCUSSION AND CONCLUSION

In the present paper, we have found the lifetime of a slightly subcritical dissipationless supercurrent state in an SNS Josephson junction due to quantum fluctuations. At low temperatures, the superconducting phase difference across the junction behaves as a massive quantum-mechanical variable, with the mass determined by the sum of the geometric (C_{geom}) and intrinsic (C_*) capacitances of the junction. While the former describes charging effects, the latter is associated with the dynamics of the Andreev bound states. We obtain that, generically, the intrinsic capacitance is expressed through the normal-state conductance G and the spectral minigap E_g induced in the normal part of the junction by the relation $C_*(\chi) = \alpha_C(\chi)(G/E_g)$, where $\alpha_C(\chi) \sim 1$ is a model-dependent coefficient. An analogous expression for C_* in the long-wire limit was conjectured in Ref. [25], based on the requirement that the RC_* time of the junction is governed by $\hbar/E_g$.

We determine the model-dependent numerical factor $\alpha_C(\chi_c)$ for two classes of Josephson junctions: an arbitrary short scatterer described by the set of its transmission eigenvalues (see Table I) and a diffusive metallic wire of arbitrary length (see Fig. 3). In the former case, we rely on the

Galaktionov-Zaikin expression (11), rederived in Appendix A. In the latter case, we employ the nonlinear σ model formalism. Depending on a particular junction, the capacitance coefficient $\alpha_C(\chi_c)$ can vary by a factor of 3, which may have a pronounced effect on the quantum tunneling rate due to its exponential sensitivity to the junction parameters. For long junctions with negligible geometric capacitance, the decay rate is determined only by G and the proximity to criticality, as given by Eq. (6). Measuring the coefficient γ in that relation and comparing it with the results (7) may provide an independent tool to determine the type of Josephson junction.

Our analysis of the quantum decay of the supercurrent is limited to low temperatures, $T \ll E_g$. For higher temperatures, finite population of quasiparticle states leads to dissipative phase dynamics, and the problem should be treated in the spirit of Ref. [3], with additional complications due to the inapplicability of the simple RSJ model, as well as possible issues with regard to inelastic relaxation and thermalization in the normal part of the junction [55].

Capacitive phase dynamics is realized only in the nondissipative supercurrent state near the criticality. Once it is destroyed by fluctuations, the junction switches to the resistive branch and the phase dynamics becomes dissipative (as for high temperatures). Though we cannot access this regime, our results can be used for a qualitative description of the hysteretic behavior observed experimentally in lateral junctions [25–28]. Assuming that an oversimplified RSJ model can qualitatively describe the SNS junction with the choice of the resistance $R \sim G^{-1}$ and capacitance $C \sim C_*$, one finds that the McCumber parameter $\beta_C = 2eI_cCR^2$ (which is the square of the quality factor $Q = \omega_pRC$) is generally of the order of 1 [formal substitution $R = G^{-1}$ and $C = C_*(\chi_c)$ yields $\beta_C = 2i_c\alpha_C(\chi_c)$, which is larger than 6 for long wires]. In this model, a large McCumber parameter, $\beta_C > 1$, is required for a hysteretic behavior. Though for long junctions the geometric capacitance is too small to explain the hysteresis, an account of the intrinsic capacitance provides sufficiently large $\beta_C \gtrsim 1$ necessary for observing a hysteretic behavior [25]. Recently, it was demonstrated that low retrapping current is a consequence of electron overheating in the normal region, when the electron temperature can be several times larger than the bath temperature [28]. The retrapping current is then identified with the critical current at the elevated electron temperature, explaining large hysteresis. By contrast, we would like to emphasize that the intrinsic junction capacitance due to the dynamics of the Andreev bound states may itself lead to hysteretic behavior, even for a perfect thermal contact with the environment.

APPENDIX A: INTRINSIC CAPACITANCE OF A SHORT JOSEPHSON JUNCTION

In this appendix, we rederive the Galaktionov-Zaikin formula (11) in the Matsubara formalism.

1. Matsubara action

An arbitrary scatterer with an energy-independent scattering matrix (short-wire limit) sandwiched between two terminals can be described by the action [50,59,60]

$$S = -\frac{1}{2}\left\langle \sum_\alpha \text{Tr}\ln\left[1 - \frac{T_\alpha}{4}(Q_L - Q_R)^2\right]\right\rangle, \qquad (A1)$$

where Q_L and Q_R are the quasiclassical Green functions in the leads. $Q(\epsilon_1, \epsilon_2)$ is a function of two energy arguments and acts as a matrix in Nambu-Gor'kov space (N). When multiplying Q matrices and taking trace, integration over ϵ_1 and ϵ_2 is assumed to be done. The time representation is defined in the conventional way as

$$Q(\tau_1, \tau_2) = \int \frac{d\epsilon_1}{2\pi} \int \frac{d\epsilon_2}{2\pi}\, Q(\epsilon_1, \epsilon_2) e^{-i\tau_1\epsilon_1 + i\tau_2\epsilon_2}. \qquad (A2)$$

In a stationary uniform superconductor, Q_S depends only on the time difference, $Q_S(\tau_1, \tau_2) = Q_S(\tau_1 - \tau_2)$, and hence it is diagonal in the energy representation, $Q_S(\epsilon_1, \epsilon_2) = 2\pi\delta(\epsilon_1 - \epsilon_2)Q_S(\epsilon_1)$, where

$$Q_S^{(\varphi)}(\epsilon) = \frac{1}{\sqrt{\epsilon^2 + \Delta^2}}\begin{pmatrix} \epsilon & \Delta e^{i\varphi} \\ \Delta e^{-i\varphi} & -\epsilon \end{pmatrix}_N, \qquad (A3)$$

and φ is superconducting phase.

Now we proceed to the derivation of the capacitive term in the action (2). We assume a zero superconducting phase on the left lead and a time-dependent phase $\chi(\tau)$ on the right lead (see Fig. 2). We assume that $\chi(\tau) = \chi_1 + x(\tau)$, where $x(\tau)$ is a small and slow function of Matsubara time τ:

$$Q_L(\tau_1, \tau_2) = Q_S^{(0)}(\tau_1 - \tau_2), \qquad (A4a)$$

$$\tilde{Q}_R(\tau_1, \tau_2) = e^{i\chi(\tau_1)\tau_z/2} Q_L(\tau_1, \tau_2) e^{-i\chi(\tau_2)\tau_z/2}, \qquad (A4b)$$

where τ_z is the Pauli matrix in the N space. Substituting these expressions into Eq. (A1), we obtain the action for the phase difference $\chi(\tau)$.

In the limit $I \to I_c$, the free-energy barrier protecting the supercurrent state $\chi = \chi_1$ is small, and one can expand the action in powers of $x(\tau) = \chi(\tau) - \chi_c$. The quadratic term can be written in the form

$$S^{(2)} = \frac{G}{4G_Q} \int \frac{d\omega}{2\pi} \int \frac{d\epsilon}{2\pi}\, x_\omega x_{-\omega}\, \mathcal{K}(\epsilon, \omega), \qquad (A5)$$

with the kernel $\mathcal{K}(\epsilon, \omega)$ calculated below. Also, the same limit guarantees that phase dynamics is slow and thus can be described (at $T = 0$) by the term $\dot\chi^2$ in Eq. (2), with the intrinsic capacitance of the junction given by

$$C_* = \pi G \int \frac{d\epsilon}{2\pi} \frac{\partial^2 \mathcal{K}(\epsilon, \omega)}{\partial \omega^2}\bigg|_{\omega=0}. \qquad (A6)$$

2. Derivation of the kernel $\mathcal{K}$

Here we calculate the kernel $\mathcal{K}(\epsilon, \omega)$ starting with the action (A1) for an arbitrary distribution of T_α (for brevity we omit brackets that denote averaging over T_α). We write $\tilde{Q}_R$ as

$$\tilde{Q}_R(\tau_1, \tau_2) = e^{ix(\tau_1)\tau_z/2} Q_R(\tau_1, \tau_2) e^{-ix(\tau_2)\tau_z/2}, \qquad (A7)$$

where $Q_R(\tau_1, \tau_2) = Q_S^{(\chi_1)}(\tau_1 - \tau_2)$ corresponds to the stationary phase χ_1. Then we insert expressions (A4) into Eq. (A1) and expand the action in powers of $x(\tau) = \chi(\tau) - \chi_1$.

The first variation of the action with respect to x reads

$$\delta^1 S = -\sum_\alpha \frac{iT_\alpha}{8}\, \text{Tr}\, \frac{\Delta\tilde{Q}}{1 - (T_\alpha/4)\Delta\tilde{Q}^2}[\tau_z\hat{x}, \tilde{Q}_R] = -\frac{i}{2}\sum_\alpha \sum_{k=0}^{\infty}\left(\frac{T_\alpha}{4}\right)^{k+1} \text{Tr}\,\Delta\tilde{Q}^{2k+1}[\tau_z\hat{x}, \tilde{Q}_R], \qquad (A8)$$

where we denote $\Delta\tilde{Q} = Q_L - \tilde{Q}_R$ and $(\hat{x})_{tt'} = x(t)\delta_{tt'}$.

In the following expression for the second variation, we will omit for simplicity terms that contain two $\hat{x}$ operators without any Q matrix in between, since such terms will give no contribution to the intrinsic capacitance. In addition, after taking the variation, we set $x(\tau) = 0$ as we are interested in the second variation near the stationary solution. The result can be written as

$$\delta^2 S = -\sum_\alpha \frac{T_\alpha}{32}\sum_{k=0}^{\infty}\left(\frac{T_\alpha}{4}\right)^k \text{Tr}\Bigg\{\sum_{l=0}^{2k-1}\Delta Q^l[\tau_z\hat{x}, Q_R]\Delta Q^{2k-l}\tau_z\hat{x}Q_R + \Delta Q^{2k}\tau_z\hat{x}Q_R\tau_z\hat{x}Q_R + \Delta Q^{2k+1}\tau_z\hat{x}Q_R\tau_z\hat{x}$$

$$-\sum_{l=1}^{2k}\Delta Q^l[\tau_z\hat{x}, Q_R]\Delta Q^{2k-l}Q_R\tau_z\hat{x} + \Delta Q^{2k+1}\tau_z\hat{x}Q_R\tau_z\hat{x} + \Delta Q^{2k}Q_R\tau_z\hat{x}Q_R\tau_z\hat{x}\Bigg\}, \qquad (A9)$$

where $\Delta Q = Q_L - Q_R$ is diagonal in the energy representation:

$$\Delta Q(\epsilon) = \frac{2\Delta\sin(\chi/2)}{\sqrt{\epsilon^2 + \Delta^2}}q, \quad q = \tau_x\sin(\chi/2) + \tau_y\cos(\chi/2). \qquad (A10)$$

Switching to the energy representation, we arrive at the action (A5) with the kernel

$$\mathcal{K}(\epsilon,\omega) = -\sum_\alpha \frac{T_\alpha G_Q}{8G} \sum_{k=0}^\infty \left(\frac{T_\alpha}{4}\right)^k \left\{ \sum_{l=0,2,...}^{2k} 2X_{2k,l}[\mathrm{tr}\, \tau_z Q_R(\epsilon)\tau_z Q_R(\epsilon') - \mathrm{tr}\, \mathbb{1}] \right.$$
$$+ \sum_{l=1,3,...}^{2k-1} X_{2k,l}[2\,\mathrm{tr}\, \tau_z Q_R(\epsilon)q\tau_z Q_R(\epsilon')q - \mathrm{tr}\, \tau_z Q_R(\epsilon)q Q_R(\epsilon)\tau_z q - \mathrm{tr}\, \tau_z Q_R(\epsilon')q Q_R(\epsilon')\tau_z q]$$
$$\left. + X_{2k,0} + X_{2k,2k} + X_{2k+1,0}\,\mathrm{tr}\, q\tau_z Q_R(\epsilon)\tau_z + X_{2k+1,2k+1}\,\mathrm{tr}\, q\tau_z Q_R(\epsilon')\tau_z \right\}, \quad (A11)$$

where $\epsilon' = \epsilon - \omega$, and

$$X_{nl} = \frac{[2\Delta \sin(\chi/2)]^n}{(\epsilon^2 + \Delta^2)^{l/2}(\epsilon'^2 + \Delta^2)^{(n-l)/2}}. \quad (A12)$$

Evaluating traces, one gets

$$\mathcal{K}(\epsilon,\omega) = -\sum_\alpha \frac{T_\alpha G_Q}{8G} \sum_{k=0}^\infty \left(\frac{T_\alpha}{4}\right)^k \left\{ -\sum_{l=0,2,...}^{2k} 4X_{2k,l}\left(1 + \frac{\Delta^2 - \epsilon\epsilon'}{\sqrt{\Delta^2+\epsilon^2}\sqrt{\Delta^2+\epsilon'^2}}\right) \right.$$
$$+ \sum_{l=1,3,...}^{2k-1} 2X_{2k,l}\left(2\frac{\epsilon\epsilon' - \Delta^2 \cos\chi}{\sqrt{\Delta^2+\epsilon^2}\sqrt{\Delta^2+\epsilon'^2}} - \frac{\epsilon^2 + \Delta^2 \cos\chi}{\epsilon^2 + \Delta^2} - \frac{\epsilon'^2 + \Delta^2 \cos\chi}{\epsilon'^2 + \Delta^2}\right)$$
$$\left. + 2X_{2k,0} + 2X_{2k,2k} + X_{2k+2,1} + X_{2k+2,2k+1} \right\}. \quad (A13)$$

Performing summation over l and k, one obtains the following for the kernel $\mathcal{K}(\epsilon,\omega)$:

$$\mathcal{K}(\epsilon,\omega) = \sum_\alpha \frac{T_\alpha G_Q}{8G} \frac{\epsilon^2 + \Delta^2}{\epsilon^2 + \Delta^2(1 - T_\alpha \sin^2\frac{\chi}{2})}\left[-4\frac{\epsilon'^2 + \Delta^2}{\epsilon^2 - \epsilon'^2}\left(1 + \frac{\Delta^2 - \epsilon\epsilon'}{\sqrt{\Delta^2+\epsilon^2}\sqrt{\Delta^2+\epsilon'^2}}\right) - 2 - \frac{\left(2\Delta \sin\frac{\chi}{2}\right)^2}{\sqrt{\Delta^2+\epsilon^2}\sqrt{\Delta^2+\epsilon'^2}} \right.$$
$$\left. + 2\frac{\sqrt{\Delta^2+\epsilon^2}\sqrt{\Delta^2+\epsilon'^2}}{\epsilon^2 - \epsilon'^2}\left(2\frac{\epsilon\epsilon' - \Delta^2 \cos\chi}{\sqrt{\Delta^2+\epsilon^2}\sqrt{\Delta^2+\epsilon'^2}} - \frac{\epsilon^2 + \Delta^2 \cos\chi}{\epsilon^2 + \Delta^2} - \frac{\epsilon'^2 + \Delta^2 \cos\chi}{\epsilon'^2 + \Delta^2}\right)\right] + \{\epsilon \leftrightarrow \epsilon'\}. \quad (A14)$$

Finally, expanding this expression to the second order in ω and integrating over ϵ, we arrive at Eq. (11) obtained by Galaktionov and Zaikin [44].

APPENDIX B: EXPLICIT FORM OF THE MATRICES V AND Ω

The rotation by the matrix V in Eq. (22) is introduced in order to get rid of the first derivative in the bulk action $S_{\mathrm{bulk}}^{(w^2)}$. Its matrix elements are given by [54]

$$V_{\epsilon_1,\epsilon_2}(r) = e^{i\Sigma_2[\zeta_{\epsilon_1}(r) + \zeta_{\epsilon_2}(r)]/2}, \quad (B1)$$

where Σ_i are Pauli matrices in the (c,d) space, and

$$\zeta_\epsilon(r) = -\int_0^r ds\, \cos\theta_\epsilon(s)\, \varphi_\epsilon(s)'. \quad (B2)$$

After such a rotation, $S_{\mathrm{bulk}}^{(w^2)}$ acquires the form (29), with the matrix potential $\Omega_{\epsilon_1,\epsilon_2}(r)$ given by (here and below we omit the spacial coordinate r for brevity) [56]

$$\Omega_{\epsilon_1,\epsilon_2} = \alpha_{\epsilon_1,\epsilon_2} - \rho_{\epsilon_1,\epsilon_2} \cos(\eta_{\epsilon_1} + \eta_{\epsilon_2})\Sigma_3$$
$$+ \rho_{\epsilon_1,\epsilon_2} \sin(\eta_{\epsilon_1} + \eta_{\epsilon_2})\Sigma_1, \quad (B3)$$

where

$$\alpha_{\epsilon_1,\epsilon_2} = \varepsilon_1 \cos\theta_{\epsilon_1} - [\theta_{\epsilon_1}'^2 + (\sin\theta_{\epsilon_1}\varphi_{\epsilon_1}')^2]/4$$
$$+ \varepsilon_2 \cos\theta_{\epsilon_2} - [\theta_{\epsilon_2}'^2 + (\sin\theta_{\epsilon_2}\varphi_{\epsilon_2}')^2]/4 \quad (B4)$$

and

$$\rho_{\epsilon_1,\epsilon_2} = \frac{1}{2}\sqrt{\theta_{\epsilon_1}'^2 + \left(\sin\theta_{\epsilon_1}\varphi_{\epsilon_1}'\right)^2}\sqrt{\theta_{\epsilon_2}'^2 + \left(\sin\theta_{\epsilon_2}\varphi_{\epsilon_2}'\right)^2}, \quad (B5)$$

and the function $\eta_\epsilon(r)$ is given by (as usual, $\varepsilon = \epsilon/E_{\mathrm{Th}}$)

$$\eta_\epsilon(r) = -2\varepsilon \int_0^r ds\, \frac{\sin^2\theta_\epsilon(s)\,\varphi_\epsilon'(s)}{\theta_\epsilon'^2(s) + [\sin\theta_\epsilon(s)\varphi_\epsilon'(s)]^2}. \quad (B6)$$

APPENDIX C: IRRELEVANCE OF NONLINEAR TERMS IN THE ACTION (23)

In this appendix, we show that for finding the capacitive effective action (2) it is sufficient to use the linear equation (31) (resulting from varying only $S_{\mathrm{bulk}}^{(w^2)}$) and neglect higher-order terms in Eq. (23).

The only possible extra contribution to the quadratic-in-x part of the action can originate from the influence of $w^{(1)}$ on the equation of motion for $w^{(2)}$ arising from cubic nonlinearity in the action (23). That would induce a correction $\delta\xi^{(2)}$ to $\xi^{(2)}$, which satisfies Eq. (31) with a nonzero right-hand side:

$$\left(-\nabla^2 + \Omega_{\epsilon_1\epsilon_2}\right)\delta\xi_{\epsilon_1\epsilon_2;i}^{(2)} = \int \frac{d\epsilon}{2\pi} \hat{h}_{\epsilon_1,\epsilon_2,\epsilon}^{ijk} \xi_{\epsilon_1,\epsilon;j}^{(1)} \xi_{\epsilon,\epsilon_2;k}^{(1)}, \quad (C1)$$

where the indices i, j, and k refer to the (c,d) space, and the operator $\hat{h}$ may contain derivatives with respect to r. The crucial point is that the boundary conditions for $\delta\xi^{(2)}$ are trivial:

$$\delta\xi^{(2)}(\pm 1/2) = 0, \qquad \text{(C2)}$$

since they are already satisfied by $\xi^{(2)}$ [see Eq. (35b)]. When substituted into $S^{(w)}$, $\delta\xi^{(2)}$ could have produced a correction to $\mathcal{K}$. But since $S^{(w)}$ [Eq. (25)] is proportional to w at the boundary, condition (C2) guarantees the absence of corrections to $\mathcal{K}$.

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