PHYSICAL REVIEW B **68**, 041306(R) (2003)

Quantum correction to the Kubo formula in closed mesoscopic systems

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(Received 29 April 2003; published 16 July 2003)

We study the energy dissipation rate in a mesoscopic system described by the parametrically driven random-matrix Hamiltonian $H[\varphi(t)]$ for the case of linear bias $\varphi = vt$. We develop a Keldysh σ-model approach to treat kinetics in such a system and use it to calculate the quantum correction to the Kubo result, which reveals the original discreteness of the energy spectrum. The first correction to the system viscosity scales $\propto v^{-2/3}$ in the orthogonal case and vanishes in the unitary case.

DOI: 10.1103/PhysRevB.68.041306 PACS number(s): 73.23.−b, 03.65.−w, 72.10.Bg

The Kubo formula[1] is one of the cornerstones of modern condensed-matter physics. It is a standard tool for calculating various linear-response functions, with conductivity as a prototypical example. Based on the lowest-order perturbation theory for a continuous spectrum, the Kubo approach connects the kinetic response of a system with respect to some external field $\varphi(t)$ to the equilibrium correlator of generalized currents.

Application of the Kubo formula essentially relies on the assumption of a continuous spectrum.[2] The spectrum may be considered as continuous if the inelastic width $\Gamma_{\rm in}$ of energy levels exceeds the system's mean level spacing Δ. In an electron system, the inelastic smearing is due to interactions (electron-electron and electron-phonon) as well as escape to reservoirs operative for open systems: $\Gamma_{\rm in}(T)=\Gamma_{\rm int}(T)+\Gamma_{\rm esc}$, where the interaction-induced smearing $\Gamma_{\rm int}(T)$ is usually a power-law function of temperature T, while $\Gamma_{\rm esc}\sim g_{\rm esc}\Delta$, with $g_{\rm esc}$ being the dimensionless conductance of the contact between the system and the leads. The widespread application of the Kubo formula for macroscopic objects is justified by the smallness of the level spacing Δ compared to the smearing $\Gamma_{\rm in}(T)$ at experimentally relevant temperatures.

However, for *closed* ($g_{\rm esc}\ll 1$) microscopic systems the condition of continuous spectrum is violated at sufficiently *low temperatures* when the interaction-induced smearing $\Gamma_{\rm int}(T)$ becomes smaller than Δ. The field had been pioneered by Gor'kov and Eliashberg[3] who studied the polarizability of a small metallic particle. Later on, dissipation with discrete specta had been extensively studied for mesoscopic rings both in the limits where the Kubo formula can (see, e.g., Refs. 4 and 5) or cannot[6] be applied. Very recently, an analogous situation was discussed in the context of vortex dynamics in layered superconductors.[7,8]

In a closed system with $\Gamma_{\rm in}(T)\ll\Delta$, the mechanism of dissipation depends on the rate $v=d\varphi/dt$ of variation of the external field $\varphi(t)$. For sufficiently slow perturbations with $v\ll v_K$ (the velocity v_K depends on the sensitivity of the spectrum to the change of φ and will be defined below), the system adiabatically follows the spectrum of its instantaneous Hamiltonian $H[\varphi(t)]$, and dissipation is due to rare Landau-Zener transitions between individual levels.[9] Fast perturbations with $v\gg v_K$ cause multiple transitions between energy levels, thereby transforming the discrete spectrum of the stationary Hamiltonian into a featureless quasicontinuous spectrum, where dissipation can be obtained with the help of the Kubo formula.

The spectrum of a generic closed quantum chaotic system of interacting electrons is properly described by the random-matrix theory,[10] provided that the relevant energy difference is smaller than the Thouless energy E_T (Ref. 11) and the system is a good conductor ($E_T\gg\Delta$).[12] For the standard Wigner-Dyson random-matrix ensembles, dissipation in the adiabatic ($v\ll v_K$) and Kubo ($v\gg v_K$) regimes had been calculated by Wilkinson.[13] In the Kubo regime, the energy dissipation rate is given by the linear-response formula,

$$W_{\rm Kubo}=\frac{\beta}{2}\pi C(0)v^2, \tag{1}$$

where $\beta=1$ for the Gaussian orthogonal (GOE) and 2 for the unitary (GUE) ensembles, and $C(0)\equiv\langle(\partial E_i/\partial\varphi)^2\rangle/\Delta^2$ is the variance of the level velocity normalized by the mean level spacing Δ,[14] which determines the critical velocity $v_K\sim\Delta/\sqrt{C(0)}$ separating the adiabatic and Kubo regimes of dissipation. (The system of units with $\hbar=1$ is used throughout the paper.) In the adiabatic limit, the dissipation rate nontrivially depends on the symmetry of the Hamiltonian:[13]

$$W_\beta=c_\beta v^{1+\beta/2}, \tag{2}$$

where $c_1=(\pi/4)\Gamma(\frac{3}{4})[C(0)]^{3/4}\Delta^{1/2}$ and $c_2=\pi C(0)$. Hence, dissipation is superohmic for GOE, while for GUE it remains ohmic, exactly coinciding with $W_{\rm Kubo}$ despite a very different mechanism of dissipation.

A schematic diagram indicating the regions of the adiabatic and Kubo regimes as functions of the inelastic width $\Gamma_{\rm in}$ and velocity v of external perturbation is shown in Fig. 1.

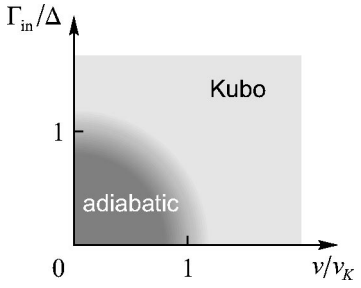

FIG. 1. Position of the adiabatic and Kubo regimes as functions of the inelastic width $\Gamma_{\rm in}$ and velocity v.

Counterintuitively, the linear-response Kubo formula does not describe the low-velocity behavior of closed systems at low temparatures when $\Gamma_{\rm in} \ll \Delta$.

In this paper, we address the question of how does discreteness of energy levels of a stationary ($v=0$) system with $\Gamma_{\rm in} \ll \Delta$ manifest itself in the Kubo regime ($v \gg v_K$) of a driven system when the levels are smeared into a quasicontinuous spectrum. To this end, we develop a Keldysh field theory which describes *both* the adiabatic and Kubo regimes of dissipation on the same ground. With its help we show that the relative correction to the high-velocity asymptotics (1) can be regularly expanded in integer powers of $(v_K/v)^{2/3}$. For the orthogonal symmetry, the first (one-loop) quantum correction to $W_{\rm Kubo}$ is given by

$$\frac{W_1}{W_{\rm Kubo}} = 1 + \frac{\Gamma\left(\frac{1}{3}\right)}{3^{2/3}} \left(\frac{v_K}{v}\right)^{2/3} + \cdots, \tag{3}$$

where the omitted terms have the $O[(v_K/v)^{4/3}]$ and the crossover velocity is defined as $v_K \equiv (2^{1/2}/\pi^2)\Delta/\sqrt{C(0)}$. Thus, the remaining correlations in the quasicontinuous spectrum enhance dissipation at $v \gg v_K$, with a gradual crossover to the superohmic regime (2) at $v \ll v_K$. In the unitary case, the first and the second (two-loop) quantum corrections to the Kubo result (1) vanish, making it tempting to conjecture that the identity $W_2 \equiv W_{\rm Kubo}$ holds for all velocities.

Quantum corrections to the quasiclassical properties of disordered systems had been the subject of intense studies in the past decades.[11,15] In treating these phenomena, the nonlinear σ model was proven to be the most powerful tool both in the perturbative and nonperturbative regimes (where it is often the only possible approach). Three versions of the σ model based on the supersymmetry,[11] replica,[16] and Keldysh[17,18] techniques had been proposed for noninteracting systems.

The problem of energy pumping by the parametrically driven Hamiltonian $H[\varphi(t)]$ belongs to the class of nonequilibrium problems, which dictates the choice of the Keldysh formalism as a solution tool. We will derive the Keldysh σ model for the parametrically driven random-matrix Hamiltonian and show that its saddle-point solution yields the kinetic equation for the distribution function, reproducing the Kubo result (1). Fluctuations near this saddle point are responsible for the quantum correction to the Kubo formula leading to Eq. (3).

We consider a time-dependent matrix Hamiltonian

$$H[\varphi(t)] = H_0 \cos k\varphi(t) + H_1 \sin k\varphi(t), \tag{4}$$

where H_0 and H_1 are the $N \times N$ matrices from the same GOE ($H^T = H$) distributed with the probability density $P(H_i) \propto \exp[-(\pi^2/4N\Delta^2)\mathrm{tr}H_i^2]$ and $k = (2\pi\sigma/\Delta)N^{-1/2}$. In the limit $N \to \infty$ assumed thereafter, Hamiltonian (4) reduces to a generic dependence $H = H_0 + V\varphi(t)$, with $V = kH_1$. The density of states for an instant Hamiltonian is given by the Wigner semicircle: $\rho(E) = \Delta^{-1}[1 - \pi^2 E^2/4N^2\Delta^2]^{1/2}$, with Δ being the mean level spacing at the center of the band. The dispersion of the matrix elements

$$\left\langle \left| \frac{\partial H_{ij}}{\partial \varphi} \right|^2 \right\rangle = \sigma^2(1+\delta_{ij}) \tag{5}$$

is independent of the matrix size N. The generalized conductance is then $C(0) = 2\sigma^2/\Delta^2$. The states of the Hamiltonian are filled by $N/2$ noninteracting fermions.

The time evolution governed by $\varphi(t)$ will change the state of the system and eventually pump some energy into it. The energy of the system will grow unless the inelastic interaction with the thermal bath is taken into account. This interaction will establish a nonequilibrium steady state. Remarkably, however, the energy dissipation rate is independent of the resulting distribution and hence can be calculated as a mean growth rate of the total system energy of noninteracting fermions.[13]

Within the Keldysh formalism, the system is described by the action (the weight is e^{-S})

$$S[\Psi] = -i \int_{-\infty}^{\infty} dt \Psi^\dagger(t) \left[i\tau_3 \frac{\partial}{\partial t} - H[\varphi(t)] \right] \sigma_3 \Psi(t), \tag{6}$$

where $\Psi(t)$ is a Grassmanian $4N$-vector field acting in the direct product of the index space of the Hamiltonian, Keldysh (K) space, and particle-hole (PH) space introduced in order to handle the orthogonal symmetry of the Hamiltonian.[11] The Pauli matrices in the K and PH spaces are denoted by σ_i and τ_i, respectively. Derivation of the σ model, which is a low-energy effective theory for action (6), is a standard procedure. One has to average $Z = \int D\Psi e^{-S}$ over Hamiltonian (4), decouple it by the 4×4 matrix $Q_{tt'}$, and integrate over fermions Ψ. Keeping the terms which are finite in the limit $N \to \infty$ and assuming linear bias $\varphi = vt$ [the case of a genetic perturbation $\varphi(t)$ is considered elsewhere[19]] we arrive at the following action for the σ model:[20]

$$S = \frac{\pi i}{2\Delta} \mathrm{Tr} \hat{E} \tau_3 Q + \frac{\pi \Omega^3}{8\Delta} \int dt\, dt' (t-t')^2 \mathrm{tr} Q_{tt'} Q_{t't}, \tag{7}$$

where $\Omega^3 \equiv \pi\sigma^2 v^2/\Delta = (\pi/2)C(0)v^2\Delta$. The first term in Eq. (7) is the standard random-matrix action,[21] which is responsible for the whole spectral statistics. The second term is of kinetic origin; it accounts for interlevel transitions of the time-dependent Hamiltonian $H[\varphi(t)]$. The field theory with action (7) describes both the adibatic and Kubo regimes of dissipation on an equal footing. It is controlled by the single parameter

$$\frac{\Omega}{\Delta} = \frac{1}{\pi} \left(\frac{v}{v_K} \right)^{2/3}, \tag{8}$$

which will be used hereafter as a measure of velocity v.

In the stationary case ($\Omega = 0$), the Keldysh Green function Q is diagonal in the energy representation:[18]

$$\Lambda = \begin{pmatrix} 1 & 2F \\ 0 & -1 \end{pmatrix} \otimes \tau_3, \tag{9}$$

where $F(E)=1-2f(E)$, and $f(E)$ is the electron distribution function. The evolution of the distribution function at $\Omega\neq 0$ is described by the saddle point of action (7). Varying with respect to the constraint $Q^2=\mathbb{1}$, one obtains the saddle-point equation $[Q,\delta S/\delta Q]=0$. Seeking the solution in form (9), we obtain the equation for the distribution function $f_{tt'}$ as

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)f_{tt'}=-\Omega^3(t-t')^2 f_{tt'}. \tag{10}$$

The same equation had been obtained in Ref. 22 starting with the diagrammatic technique. Performing the Wigner transformation $f(E,t)=\int d\tau e^{iE\tau}f_{t+\tau/2,t-\tau/2}$ we arrive at the kinetic equation

$$\frac{\partial f(E,t)}{\partial t}=\Omega^3\frac{\partial^2 f(E,t)}{\partial E^2}. \tag{11}$$

This is a diffusion equation in the energy space, with Ω^3 being the bare "diffusion coefficient." The rate of energy pumping for the system described by the kinetic equation (11) is given by

$$W=\int E\frac{\partial f(E,t)}{\partial t}\frac{dE}{\Delta}=\frac{\Omega^3}{\Delta}, \tag{12}$$

which coincides with the result of the Kubo formula (1).

The kinetic equation (11) is a true saddle point of the action for all velocities v. Solution (12) for the dissipation rate predicted is valid, however, only in the Kubo regime, and is completely incorrect in the adiabatic regime. The reason is that the saddle-point approximation is justified by the large value of the parameter Ω/Δ. In the adiabatic regime, the saddle-point approximation fails and one has to take all the Q manifolds into account. As a result of this procedure, solution (2) should be reproduced. Note the interchange of steps with respect to Wilkinson's derivation:[13] He first calculates the probability of the Landau-Zener transition and then averages it over the distribution of avoided crossings. Here we, instead, first average over randomness and then extract the dissipation rate from the field theory (7). Thus, it is a challenging problem to demonstrate how the adiabatic result (2) should be obtained from the field theory (7).

Quantum correction to the Kubo result (12) in the limit $\Omega/\Delta\gg 1$ (Δ/Ω is the loop expansion parameter) can be obtained in the regular way by expanding over Gaussian fluctuations near the saddle point. The matrix Q can be written as[18]

$$Q=U_F^{-1}PU_F,\quad U_F=\begin{pmatrix}1 & F\\ 0 & -1\end{pmatrix}, \tag{13}$$

with the Hermitian matrix P having an additional symmetry $P^T=\sigma_1\tau_2 P\tau_2\sigma_1$ imposed by the orthogonal symmetry of the Hamiltonian. We choose the standard rational parametrization of the P matrix, $P=\sigma_3\tau_3(1+V/2)(1-V/2)^{-1}$, with the unit Jacobian. The matrix V is explicitly given by

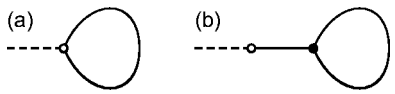

FIG. 2. One-loop corrections to the system energy. The solid lines stand for propagators (15) and (16), the dashed line denotes for the source field $\kappa(t)$, and the open and solid vertices come from the terms S_κ and S, respectively.

$$V=\begin{pmatrix}0 & a & b & 0\\ -a^\dagger & 0 & 0 & -b^T\\ -b^\dagger & 0 & 0 & a^T\\ 0 & b^* & -a^* & 0\end{pmatrix}, \tag{14}$$

where the inner (outer) grading corresponds to the PH (K) space. The elements $a_{tt'}$ and $b_{tt'}$ describe Cooperon and diffuson modes, respectively. Their bare propagators read[22–25]

$$\langle a_{t_1t_2}a^*_{t_3t_4}\rangle=\frac{2\Delta}{\pi}\delta(t_{13}+t_{24})\theta(t_{13})e^{-\Omega^3(t_{13}^3/3+t_{13}t_{14}^2)}, \tag{15}$$

$$\langle b_{t_1t_2}b^*_{t_3t_4}\rangle=\frac{2\Delta}{\pi}\delta(t_{13}-t_{24})\theta(t_{13})e^{-\Omega^3 t_{13}t_{12}^2}, \tag{16}$$

where $t_{ij}\equiv t_i-t_j$. In the stationary case ($\Omega=0$), Eqs. (15) and (16) describe the standard cooperon and diffuson in the time domain. The exponential decay of the correlators at the time scale Ω^{-1} is a manifestation of dephasing by the time-dependent perturbation.[22,23]

The system energy $\langle E(t)\rangle$ (apart from an additive constant) can then be obtained as a functional derivative with respect to the quantum source field $\kappa(t)$:

$$\langle E(t)\rangle=\frac{1}{2}\frac{\delta Z[\kappa]}{\delta\kappa(t)}\Big|_{\kappa=0},\quad Z[\kappa]=\int e^{-S-S_\kappa}DQ, \tag{17}$$

where the source action $S_\kappa=(\pi/2\Delta)\mathrm{Tr}\kappa\hat{E}\sigma_1\tau_3 Q$.

At the saddle point, Eq. (17) reproduces result (12). The one-loop diagrams are shown in Fig. 2. The diagram 2(a) obtained from the expansion of the source term S_κ contains either $\langle a(t_1,t_2)a^*(t_1,t_3)\rangle$ or $\langle b(t_1,t_2)b^*(t_1,t_3)\rangle$ which are proportional to $\theta(0)$. In the Keldysh formalism, the Heaviside θ function of zero argument evaluates to zero, which is related with the causality of the theory.[21] Calculating for Fig. 2(b) we obtain for the quantum correction to the dissipation rate

$$\frac{\delta W_1}{W_{\mathrm{Kubo}}}=\frac{\Gamma\left(\frac{1}{3}\right)}{3^{2/3}\pi}\frac{\Delta}{\Omega}, \tag{18}$$

which after rewriting in terms of velocity leads to Eq. (3). This result is applicable also in the regime of a smeared spectrum, provided that $\Omega\tau_\varphi\gg 1$, where τ_φ is the phase-coherence time of a stationary system. Under this condition, the interaction-induced dephasing can be neglected compared to that due to a time-dependent perturbation.

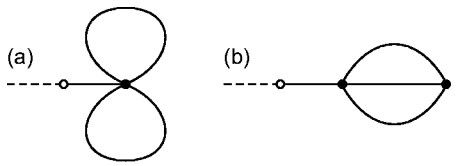

FIG. 3. Two-loop corrections to $\langle E(t)\rangle$ in the unitary case.

In the unitary case, the diagram of Fig. 2(b) vanishes indicating the absence of the one-loop quantum correction to the Kubo result. In the two-loop approximation, only the diagrams shown in Fig. 3 contribute to the dissipation rate for the GUE. Each of them has the order of $(\Delta/\Omega)^2$, but their sum is zero. Thus, for the case of the unitary symmetry, the two-loop correction also vanishes. Taking into account the coincidence between the low- and high-velocity asymptotics (1) and (2) for $\beta=2$, such a cancellation is a strong argument in favor of the exact identity $W_2=W_{\mathrm{Kubo}}$ valid for all velocities. At present we cannot prove this conjecture, but we hope that this can be done by an accurate analysis of the σ model (7). We conjecture that the independence of dissipation on the velocity v might be another manifestation of the peculiar properties of the unitary group.[26] It is worth mentioning that the direct quantum-mechanical calculation of the transition rates at $v\sim v_K$ *prior* to disorder averaging seems completely impossible. Therefore, the identity $W_2=W_{\mathrm{Kubo}}$ for the averaged dissipation rate would be a highly nontrivial fact.

The results obtained are relevant for the description of heating effect in closed 1-μm-size quantum dots below 0.3 K (Ref. 27) when the phase-coherence time $\tau_\varphi\gtrsim\Delta^{-1}$. Vortex motion in impure superconductors is another field of application, where the conjecture $W_2=W_{\mathrm{Kubo}}$ would indicate that the dissipative flux-flow conductivity is independent of the velocity of vortex motion.[7]

Summarizing, we have developed the Keldysh σ-model approach for studing energy pumping in parametrically driven random-matrix ensembles, thereby opening the way to an analytical treatment of quantum interference effects in driven mesoscopic systems. With its help we calculated the leading correction to the high-velocity dissipation, which reveals the original discreteness of the spectrum of the stationary Hamiltonian. This correction emerges in the one-loop approximation for the GOE and is absent within the two-loop accuracy for the GUE.

I am grateful to D. M. Basko, Ya. M. Blanter, D. A. Ivanov, M. V. Feigel'man, V. E. Kravtsov, A. I. Larkin, and Yu. V. Nazarov for illuminating discussions. Financial support from the SCOPES, NWO, RFBR under Grant No. 01-02-17759, the Russian Ministry of Science, the Russian Science Support Foundation, the Dynasty Foundation, and the ICFPM is acknowledged.

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