

## Keldysh action for disordered superconductors

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The Keldysh representation of the functional integral for the interacting electron system with disorder is used to derive microscopically an effective action for dirty superconductors. In the most general case this action is a functional of the  $8 \times 8$  matrix  $Q(t, t')$  which depends on two time variables, and on the fluctuating order parameter field and electric potential. We show that this approach reproduces, without the use of the replica trick, the well-known result for the Coulomb-induced renormalization of the electron-electron coupling constant in the Cooper channel. Turning to the results, we calculate the effects of the Coulomb interaction upon (i) the subgap Andreev conductance between the superconductor and the two-dimensional dirty normal metal, and (ii) the Josephson proximity coupling between superconductive islands via such a metal. These quantities are shown to be strongly suppressed by the Coulomb interaction at sufficiently low temperatures due to *both* zero-bias anomaly in the density of states and disorder-enhanced repulsion in the Cooper channel.

### I. INTRODUCTION

Electron transport in hybrid superconductive-normal systems at low temperatures is governed by the Andreev reflection. Both finite-voltage conductance  $G_A$  between superconductive and normal electrodes, and Josephson critical current  $I_c$  between two superconductive banks, separated by the normal region, are determined by the Cooper pair propagation in the normal metal. The theory of the Andreev conductance (without Coulomb effects) was developed in, e.g., Refs. 1–3, whereas the Josephson coupling was calculated (with the account of short-range electron interaction) in Ref. 4. When the normal conducting region is made of a dirty metal film, or two-dimensional (2D) electron gas with a low density of electrons, Coulomb interaction in the normal region may lead to strong quantum fluctuations which suppress both the Andreev conductance and the Josephson proximity effect. Several different kinds of quantum effects are known to be relevant in low-dimensional conductors at low temperatures. Quantum corrections to the conductivity of two-dimensional dirty conductors grow logarithmically as temperature  $T$  decreases and become large at  $\ln(1/T\tau) \sim g$  (where  $g = (\hbar/e^2)\sigma$  is the dimensionless conductance,  $\sigma$  is the conductance per square, and  $\tau$  is the elastic scattering time). There are two main types of these effects: weak localization corrections,<sup>5,6</sup> and interaction-induced corrections.<sup>7</sup> Other important quantum effects include interaction-induced suppression of the tunneling conductance (“zero-bias anomaly”<sup>7,8</sup>), and disorder-induced suppression<sup>9–13</sup> of superconductive transition temperature  $T_c$ . These effects are of the relative order of  $g^{-1} \ln^2(1/T\tau)$ , i.e., much stronger than the weak localization and interaction effects. Earlier studies of all those effects<sup>6,7,9,10</sup> employed perturbative diagram

technique based on the expansion over the semiclassical parameter  $(E_F\tau)^{-1} \sim g^{-1}$ . This method has an obvious drawback: the number and complexity of diagrams grows fast with the order of perturbation theory, which makes its use very difficult in the lowest-temperature region where effects are strong. Some combination of the perturbative diagram technique and functional methods was used in an earlier paper,<sup>14</sup> where an attempt to calculate the effect of long-range Coulomb interaction upon  $T_c$  was made. More advanced functional methods were then developed in the weak localization theory,<sup>15,16</sup> which made it possible to average the fermionic functional integral over disorder, and reduce it to an “effective” form which contains slowly varying diffusive modes only. Those approaches have used either the replica trick<sup>15</sup> or the method of supersymmetry.<sup>16</sup> Whereas the supersymmetry method was found to be very powerful and convenient for the study of single-electron effects, it cannot be used in the cases where quantum corrections due to electron-electron interactions are important. The replica method was generalized for the interacting systems by Finkelstein (cf. Ref. 12 for the review); in particular, he has shown that in dirty films the superconductive  $T_c$  vanishes at  $g \sim \ln^2(1/T_{c0}\tau)$  [here  $T_{c0}$  is the bare (BCS) transition temperature]. The drawback of the replica method is that it contains an unphysical procedure of analytic continuation over the number of replicas  $n \rightarrow 0$ , and, also, it is difficult to use for the study of nonequilibrium phenomena.

A long time ago Keldysh<sup>17</sup> proposed an approach which allows one to treat kinetic phenomena in metals with the use, both, of the Green’s-function technique, and of the kinetic equation for the distribution function. This approach was found to be especially fruitful in the theory of superconductivity, where dynamic equations for the Green’s functions

were derived in the dirty limit<sup>18</sup> (cf. also review articles, Refs. 19,20). In the static limit, these equations coincide with the Usadel equations.<sup>21</sup> The Keldysh approach often was found to be the most simple and transparent, even for the treatment of linear-response problems, since it does not involve tedious analytic continuation procedures. It is also the only known method for the treatment of nonlinear and/or nonequilibrium problems. In some cases nonlinearities with respect to both external and *fluctuating* fields are important, so one needs either to sum up very large number of diagrams, or to develop some effective action formalism within the Keldysh approach. Such an approach was recently developed, for normal metals, by Kamenev and Andreev.<sup>22</sup> In many respects we will follow this seminal paper in our analysis. A similar approach was also recently developed in Ref. 23, where Finkelstein's renormalization group equations for dirty metals were rederived for the case of short-range electron-electron interaction. For the earlier approaches to develop functional integral methods in the Keldysh representation see Refs. 24–27.

In the present paper we will develop the Keldysh functional approach for dirty superconductors, and will use it for the study of the Coulomb-interaction effects in the low-temperature Andreev conductance  $G_A$  between superconductor and normal metal and in the Josephson proximity coupling  $E_J$  via dirty 2D metal. It will be shown that both of the above-mentioned effects, interaction-induced suppression of the tunneling density of states (DOS), and renormalization of the Cooper-channel interaction, contribute considerably into the suppression of  $G_A$  and  $E_J$  in the low-energy limit. These two effects differ in the following sense: the specific form of the DOS suppression depends crucially on the long-range behavior of the Coulomb potential (and thus can be varied externally by changing electromagnetic environment), whereas renormalization of the Cooper-channel interaction depends on the short-distance Coulomb amplitude only. If long-range Coulomb forces are suppressed (i.e., by placing a nearby screening electrode), the DOS suppression effect may become weak. In this case the main effect comes from the presence of short-range repulsion in the Cooper channel; as a result, both the Andreev conductance  $G_A(\omega)$  and the Josephson proximity coupling  $E_J(r)$  exhibit anomalous power-law suppression in the infrared limit, with exponents of the order of  $g^{-1/2}$ . In the case of no static Coulomb screening, the DOS suppression effect is the strongest one in the asymptotic infrared limit, and it leads to the ‘‘log-normal’’ suppression of  $G_A(\omega)$  and  $E_J(r)$  as  $(\omega, D/r^2) \rightarrow 0$ . The influence of long-range Coulomb interaction on the Andreev conductance was treated previously<sup>28</sup> in a kind of phenomenological circuit theory. In the asymptotic limit  $(1/g)\ln^2(1/\omega\tau) \gg 1$  our results are in agreement with those of Ref. 28 and provide a microscopic derivation for the effective impedance function used there phenomenologically; in the intermediate region  $(1/g)\ln^2(1/\omega\tau) \sim 1$  an additional contribution due to the Cooper-channel repulsion (not treated in Ref. 28) is shown to be equally important.

The rest of the paper is organized as follows: in Sec. II we describe the formalism of the Keldysh-type functional integral and derive an effective action resulting from disorder averaging; in Sec. III we determine the saddle manifold of the above action and derive a kind of a nonlinear  $\sigma$ -model

action formulated in terms of an  $8 \times 8$  matrix  $Q$  which depends on two times and one spacial coordinate, and of fluctuating order parameter and electromagnetic fields. In Sec. IV the basics of perturbation theory and diagram technique for the derived  $\sigma$  model are presented. In Sec. V, in order to demonstrate the technique developed, we rederive Finkelstein's renormalization-group equations and calculate  $T_c$  suppression for dirty superconductive films. Section VI is devoted to the calculation of the Andreev conductance as a function of frequency and/or temperature in the presence of the Coulomb effects; we consider two different geometries of normal-metal–superconductor (N-S) contact both in the presence and absence of the DOS suppression effect. In Sec. VII we switch to the calculation of the Josephson proximity coupling, for the same two geometrical configurations. Section VIII contains discussion and conclusions. Finally, some technical details are presented in two Appendixes.

## II. DERIVATION OF THE EFFECTIVE ACTION IN THE KELDYSH FORM

The Lagrangian of the electron system interacting via electromagnetic field and subject to disorder potential can be written as

$$\mathcal{L} = \mathcal{L}^e + \mathcal{L}^f, \quad (2.1)$$

where

$$\begin{aligned} \mathcal{L}^e = \int \left( \psi_\alpha^* \left[ i \frac{\partial}{\partial t} + \frac{(\nabla - i\mathbf{a})^2}{2m} + \mu - U_{\text{dis}}(\mathbf{r}) + \phi \right] \psi_\alpha \right. \\ \left. + \Delta \psi_\uparrow^* \psi_\downarrow^* + \Delta^* \psi_\downarrow \psi_\uparrow \right) d\mathbf{r} \end{aligned} \quad (2.2)$$

and

$$\mathcal{L}^f = \int \frac{\mathbf{E}^2 - \mathbf{H}^2}{8\pi e^2} d^3\mathbf{r} + \frac{\nu}{\lambda} \int \Delta^* \Delta d\mathbf{r} \quad (2.3)$$

describe electron and field contributions, respectively. Here  $\psi_\alpha(\mathbf{r}, t)$  is a Grassmann spinor field ( $\alpha = \uparrow, \downarrow$ ),  $\phi(\mathbf{r}, t)$  and  $\mathbf{a}(\mathbf{r}, t)$  are the electromagnetic potentials. Space integrals in Eq. (2.2) and in the second term of Eq. (2.3) are taken over the system considered, while the free electromagnetic Lagrangian given by the first term of Eq. (2.3) contains integration over the whole three-dimensional space.  $\mu$  is the chemical potential, and  $\nu$  is the density of states per one spin projection at the Fermi level. A short-range interaction in the Cooper channel mediated by electron-electron and electron-phonon interactions with momentum transfer of the order of  $p_F$  is decoupled in the standard way<sup>29</sup> by the  $\Delta$  field. Our superconductive coupling constant  $\lambda$  coincides with the Finkelstein's definition<sup>12</sup> of  $\Gamma_c$ . Generally speaking, one should also introduce singlet and triplet coupling constants,  $\Gamma$  and  $\Gamma_2$  in Finkelstein's notations, in the diffusion channel. In two-dimensional systems, the correction to them has the relative order of the weak localization effect,  $g^{-1}\ln(1/\Omega\tau)$ , with  $\Omega$  being the relevant energy scale. Therefore, for the study of the Coulomb effects in the Cooper channel and in the tunneling density of states, they may be considered con-

stant and can be incorporated into the Fermi-liquid renormalization of the parameters of the Lagrangian (2.2). Through the paper  $\hbar = c = 1$ .

We are going to construct an effective theory of soft modes in the problem. For this purpose one has to be able to take all possible channels into account that is accomplished<sup>16</sup> by introducing a bispinor  $\Psi = 1/\sqrt{2}(\psi_\uparrow, \psi_\downarrow, \psi_\downarrow^*, -\psi_\uparrow^*)^T$ .  $\Psi$  is a vector in the four-dimensional space  $\Omega$  which can be considered as the direct product  $S \otimes T$  of the spin ( $\psi_\uparrow, \psi_\downarrow$ ) and time-reversal ( $\psi, \psi^*$ ) spaces. The correlations between different time-reversal components of  $\Psi$  are responsible for the quantum correction to the conductivity in the orthogonal case<sup>6</sup> when the Hamiltonian possesses time-reversal symmetry. On the other hand, in studying superconducting phenomena, it is convenient to rearrange components of the bispinor in a different manner, separating explicitly the Gor'kov-Nambu<sup>30,31</sup> ( $\psi_\uparrow, \psi_\downarrow^*$ ) and spin spaces. Finally, one can think of  $\Psi$  as acting in the direct product of the Nambu ( $N$ ) and time-reversal spaces. These three representations are equivalent,  $\Omega \simeq S \otimes T \simeq N \otimes S \simeq N \otimes T$ ; we will not specify a certain one and will change between them depending on the problem at hand. For later reference, we define the Pauli matrices in the spin, time-reversal, and Nambu spaces as  $s_i, t_i, \tau_i$ , respectively ( $i=0,1,2,3$ ). The action of these matrices on the bispinor  $\Psi$  is equivalent to multiplication by  $4 \times 4$  matrices; a few examples of them are listed below:

$$\begin{aligned} s_x &\simeq \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \\ \tau_x &\simeq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ t_y &\simeq \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.4)$$

Below we will use the vector  $\Psi$  together with its conjugate vector  $\Psi^+$ . These vectors are linearly dependent and related by  $\Psi^+ = (C\Psi)^T = 1/\sqrt{2}(\psi_\uparrow^*, \psi_\downarrow^*, -\psi_\downarrow, \psi_\uparrow)$ , where the charge-conjugation matrix  $C = it_y \otimes s_0 \equiv i\tau_y \otimes s_x$ . In terms of the  $\Psi$  field, the electron Lagrangian  $\mathcal{L}^e$  can be rewritten as

$$\begin{aligned} \mathcal{L}^e = \int d\mathbf{r} \Psi^+ \left[ i\Xi \frac{\partial}{\partial t} + \frac{(\Xi \nabla - i\mathbf{a})^2}{2m} \right. \\ \left. + \mu - U_{\text{dis}}(\mathbf{r}) + \phi + \hat{\Delta} \right] \Psi. \end{aligned} \quad (2.5)$$

Here  $\hat{\Delta} = i\tau_y \text{Re} \Delta + i\tau_x \text{Im} \Delta = \tau_+ \Delta - \tau_- \Delta^* = |\Delta| [\tau_+ e^{i\theta} - \tau_- e^{-i\theta}]$ ,  $\theta$  is the phase of the order parameter,  $\tau_\pm = (\tau_x \pm i\tau_y)/2$ , and the  $4 \times 4$  matrix  $\Xi$  is given by

$$\Xi = t_z \otimes s_0 \equiv \tau_z \otimes s_0. \quad (2.6)$$

Within the Keldysh approach the time evolution of the system is considered along the Keldysh contour  $\mathcal{C}$  going from  $t = -\infty$  to  $t = +\infty$  and then back to  $-\infty$ . At the initial time,  $t = -\infty$ , the system is supposed to be in the thermal equilibrium, with the interaction and disorder potential being turned off. The latter are switched on adiabatically during the time evolution. The electromagnetic potentials  $\phi$  and  $\mathbf{a}$  entering the action (2.8) consist of fluctuating and external (source) parts:  $\phi = \phi_{fl} + \phi_s$ ,  $\mathbf{a} = \mathbf{a}_{fl} + \mathbf{a}_s$ . The partition function describing the evolution along the contour can be written in terms of the functional integral over fermionic fields  $\psi$  and  $\psi^*$  and fluctuating electromagnetic field potentials  $\phi_{fl}$ ,  $\mathbf{a}_{fl}$  as

$$Z = \int D\{\psi^*, \psi\} D\{\phi_{fl}, \mathbf{a}_{fl}\} D\{\Delta\} e^{iS}, \quad (2.7)$$

where the action is given by

$$S = \int_{\mathcal{C}} \mathcal{L} dt. \quad (2.8)$$

External fields (introduced through the *source terms* in the action) should not be integrated out in the functional integral (2.7). If all external fields are *classical*, i.e., identical for the forward and backward propagation, then the evolution along the Keldysh contour brings the system back to the initial state. In this case the partition function is automatically normalized to unity,  $Z = 1$ . To get a nontrivial result for physical quantities one has to consider the generating functional for the source fields having *quantum* components which are different on the upper and lower parts of the contour. We will discuss the role of the source terms in Sec. VI B and will operate meanwhile with the partition function given by Eq. (2.7).

Next we divide each dynamical field into two parts residing on the forward and backward branches of the contour and labeled by the indices 1 and 2, respectively, and combine them into two vectors in the Keldysh space:

$$\begin{aligned} \vec{\Psi} &= \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad \vec{\mathbf{a}} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}, \\ \vec{\Delta} &= \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}, \quad \vec{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \end{aligned} \quad (2.9)$$

Then the action can be written as  $S = \int_{-\infty}^{\infty} \mathcal{L} dt$  with the Lagrangian given by  $L = \mathcal{L}_1 - \mathcal{L}_2$ , where  $\mathcal{L}_i \equiv \mathcal{L}[\psi_i^*, \psi_i, \phi_i, \mathbf{a}_i, \Delta_i]$ . It is convenient to arrange the components of the Bose fields  $\vec{\phi}$ ,  $\vec{\mathbf{a}}$ ,  $\vec{\Delta}$ , and  $\vec{\theta}$  in the matrix form:

$$\begin{aligned} \vec{\phi} &= \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}, \quad \vec{\mathbf{a}} = \begin{pmatrix} \mathbf{a}_1 & 0 \\ 0 & \mathbf{a}_2 \end{pmatrix}, \\ \vec{\Delta} &= \begin{pmatrix} \hat{\Delta}_1 & 0 \\ 0 & \hat{\Delta}_2 \end{pmatrix}, \quad \vec{\theta} = \begin{pmatrix} \hat{\theta}_1 & 0 \\ 0 & \hat{\theta}_2 \end{pmatrix}. \end{aligned} \quad (2.10)$$

Making use of the Keldysh-space matrices (2.10) we can rewrite the electron and field parts of the Lagrangian in the following concise form:

$$L^e = \int d\mathbf{r} \bar{\Psi}^+ \left[ i\Xi \frac{\partial}{\partial t} + \frac{(\Xi \nabla - i\vec{\mathbf{a}})^2}{2m} + \mu - U_{\text{dis}}(\mathbf{r}) + \vec{\phi} + \vec{\Delta} \right] \sigma_z \bar{\Psi}, \quad (2.11)$$

$$L^f = \int \frac{\vec{\mathbf{E}}^T \sigma_z \vec{\mathbf{E}} - \vec{\mathbf{H}}^T \sigma_z \vec{\mathbf{H}}}{8\pi e^2} d^3\mathbf{r} + \frac{\nu}{\lambda} \int \vec{\Delta}^+ \sigma_z \vec{\Delta} d\mathbf{r}, \quad (2.12)$$

where  $\vec{\mathbf{E}}$  and  $\vec{\mathbf{H}}$  are expressed in terms of  $\vec{\phi}$  and  $\vec{\mathbf{a}}$  in the standard way, and  $\sigma_i$  denote the Pauli matrices in the Keldysh space.

Now we are in a position to perform disorder averaging. The Keldysh formalism allows us to average the partition function directly utilizing its independence of realization of disorder potential. For the latter we will assume the model of a Gaussian  $\delta$ -correlated white noise with the variance

$$\langle U_{\text{dis}}(\mathbf{r}) U_{\text{dis}}(\mathbf{r}') \rangle = \frac{\delta(\mathbf{r} - \mathbf{r}')}{2\pi\nu\tau}. \quad (2.13)$$

Integrating out disorder potential generates a four-fermion term in the action:

$$S_{\text{dis}} = \frac{i}{4\pi\nu\tau} \int d\mathbf{r} dt dt' [\bar{\Psi}^+(\mathbf{r}, t) \sigma_z \bar{\Psi}(\mathbf{r}, t)] \times [\bar{\Psi}^+(\mathbf{r}, t') \sigma_z \bar{\Psi}(\mathbf{r}, t')]. \quad (2.14)$$

The slow part of the resulting nonlocal in time action can be decoupled in the standard way (cf. Ref. 16) by the Hubbard-Stratonovich matrix field  $\check{Q}$ :

$$e^{iS_{\text{dis}}} = \int D\check{Q} \exp \left\{ -\frac{1}{2\tau} \int d\mathbf{r} dt dt' \bar{\Psi}^+(\mathbf{r}, t) \times \check{Q}(\mathbf{r}, t, t') \sigma_z \bar{\Psi}(\mathbf{r}, t') - \frac{\pi\nu}{8\tau} \int d\mathbf{r} dt dt' \text{tr} \times \check{Q}(\mathbf{r}, t, t') \check{Q}(\mathbf{r}, t', t) \right\}. \quad (2.15)$$

In the Keldysh formalism,  $\check{Q}(\mathbf{r}, t, t')$  is a matrix in the time space as well as an  $8 \times 8$  matrix in the  $K \otimes \Omega$  space; it is local in the coordinate space once the  $\delta$ -correlated random potential is considered. Here and in what follows  $\text{tr}(\dots)$  stands for the trace in the  $K \otimes \Omega$  space whereas the complete operator trace involving integration over space and time indices will be denoted by  $\text{Tr}(\dots)$ .

After the Hubbard-Stratonovich transformation (2.15), the fermionic part of the action becomes quadratic and can be written as  $S^e = \text{Tr} \bar{\Psi}^+ G^{-1} \bar{\Psi}$ , where the inverse Green function is defined as

$$G^{-1} = \left[ i\Xi \frac{\partial}{\partial t} + \frac{(\Xi \nabla - i\vec{\mathbf{a}})^2}{2m} + \mu + \frac{i}{2\tau} \check{Q} + \vec{\phi} + \vec{\Delta} \right] \sigma_z. \quad (2.16)$$

The field part of the action takes the form

$$S^f = \text{Tr} \frac{\vec{\mathbf{E}}^T \sigma_z \vec{\mathbf{E}} - \vec{\mathbf{H}}^T \sigma_z \vec{\mathbf{H}}}{8\pi e^2} + \frac{\nu}{\lambda} \text{Tr} \vec{\Delta}^+ \sigma_z \vec{\Delta} + \frac{i\pi\nu}{8\tau} \text{Tr} \check{Q}^2. \quad (2.17)$$

As mentioned above, the trace operator  $\text{Tr}(\dots)$  includes integration over space coordinates; in the first term of Eq. (2.17) this integral goes over the whole three-dimensional space, whereas in the other terms the integral is taken over the volume of the system considered. In the present paper we will consider thin metallic films only, so the integral in the second and third terms will be effectively two-dimensional. Gaussian integration over  $\bar{\Psi}$  can easily be performed resulting in

$$S^e = -\frac{i}{2} \text{Tr} \ln G^{-1}. \quad (2.18)$$

Among four Keldysh subblocks of the Green function (2.16) only three appear to be linearly independent<sup>17</sup>. To simplify its structure, it is convenient to pass to the rotated basis:<sup>18</sup>

$$G' = L \sigma_z G L^{-1}, \quad L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (2.19)$$

After such a rotation  $G'$  acquires a triangular form (provided that source terms have no quantum component):

$$G' = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}, \quad (2.20)$$

where  $G^{R(A)}(t, t') = 0$  for  $t \leq t'$  ( $t \geq t'$ ).

Apart from the Green function, it is also convenient to make a similar transformation for  $\check{Q}$ :

$$\check{Q}' = L \check{Q} L^{-1}, \quad (2.21)$$

and to rotate all two vectors defined in Eq. (2.9) according to

$$\vec{\phi}' \equiv \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi_1 + \phi_2 \\ \phi_1 - \phi_2 \end{pmatrix}, \quad (2.22)$$

and analogously for  $\vec{\mathbf{a}}$ ,  $\vec{\Delta}$ , and  $\vec{\theta}$ . For the reasons discussed above,  $\phi'_1$  and  $\phi'_2$  will be referred to as the classical and quantum components of the field. The matrices (2.10) will transform according to

$$\vec{\phi}' = L \vec{\phi} L^{-1} = \begin{pmatrix} \phi'_1 & \phi'_2 \\ \phi'_2 & \phi'_1 \end{pmatrix} \equiv \phi'_i \gamma^i \quad (2.23)$$

(and analogously for  $\vec{\mathbf{a}}$ ,  $\vec{\Delta}$ , and  $\vec{\theta}$ ), where, following Kamelev and Andreev,<sup>22</sup> we introduced two vertex matrices

$$\gamma^1 \equiv \sigma_0, \quad \gamma^2 \equiv \sigma_x. \quad (2.24)$$

In some cases (e.g., when one considers a uniform superconductor on time scales much longer than the inverse gap) it is sufficient to treat the absolute value  $|\Delta|$  of the superconductive order parameter as a constant, while taking into account fluctuations of its phase. Then, in the rotated basis, the

expression for  $\vec{\Delta}'$  can be written as  $\vec{\Delta}' = |\Delta|[\tau_+ e^{i\tilde{\theta}'} - \tau_- e^{-i\tilde{\theta}'}]$ , or, in terms of the classical,  $\theta'_1$ , and quantum,  $\theta'_2$ , components:

$$\hat{\Delta}'_1 = |\Delta|[\tau_+ e^{i\theta'_1} - \tau_- e^{-i\theta'_1}] \cos \theta'_2, \quad (2.25a)$$

$$\hat{\Delta}'_2 = |\Delta|[\tau_+ e^{i\theta'_1} + \tau_- e^{-i\theta'_1}] i \sin \theta'_2. \quad (2.25b)$$

As a result, the Green function can be written as

$$G'^{-1} = i\Xi \frac{\partial}{\partial t} + \frac{(\Xi \nabla - i\vec{\mathbf{a}}')^2}{2m} + \mu + \frac{i}{2\tau} \check{Q}' + \vec{\phi}' + \vec{\Delta}', \quad (2.26)$$

and the action takes the form

$$S = -\frac{i}{2} \text{Tr} \ln G'^{-1} + \text{Tr} \frac{\vec{\mathbf{E}}'^T \sigma_x \vec{\mathbf{E}}' - \vec{\mathbf{H}}'^T \sigma_x \vec{\mathbf{H}}'}{4\pi e^2} + \frac{2\nu}{\lambda} \text{Tr} \vec{\Delta}' + \sigma_x \vec{\Delta}' + i \frac{\pi\nu}{8\tau} \text{Tr} \check{Q}'^2. \quad (2.27)$$

The factor of 2 difference between the coefficients in the terms containing  $\vec{\mathbf{E}}$ ,  $\vec{\mathbf{H}}$ , and  $\vec{\Delta}$  in Eqs. (2.17) and (2.27) is due to the Jacobian of the transformation (2.22). In what follows we will omit the prime at the designation of the Keldysh-rotated fields. This cannot lead to an ambiguity since the original basis will not be used in the subsequent analysis.

### III. $\sigma$ MODEL

In this section we will construct an effective theory which describes low-energy physics of the action (2.27). To start, we subject the action to the stationary phase analysis. It is a functional of the matrix field  $\check{Q}$  and the bosonic fields  $\vec{\phi}$ ,  $\vec{\mathbf{a}}$ ,  $\vec{\Delta}$ , and one has to vary the action with respect to all of them in order to get a set of the saddle-point equations. First of all we note that quantum components of the bosonic fields are equal to zero in the mean-field approximation. Thus, in this section, we will designate their classical components without the subscript ‘‘1’’ for brevity (i.e.,  $\phi \equiv \phi_1$ , etc.). Below we will use  $\check{Q}$  matrices defined in the energy domain according to the relation

$$\check{Q}_{\epsilon\epsilon'} = \iint dt dt' e^{i\epsilon t - i\epsilon' t'} \check{Q}_{tt'}. \quad (3.1)$$

Varying with respect to  $\check{Q}$  yields the saddle-point equation

$$\check{Q}(\mathbf{r}) = \frac{i}{\pi\nu} G(\mathbf{r}, \mathbf{r}). \quad (3.2)$$

In the absence of quantum components, the Green function has a triangular form (2.20) and so does  $\check{Q}$ :

$$\check{Q} = \begin{pmatrix} Q^R & Q^K \\ 0 & Q^A \end{pmatrix}_K, \quad (3.3)$$

where  $Q^R$ ,  $Q^A$ , and  $Q^K$  are matrices in the space  $\Omega$ . In the stationary case, the solution  $\check{Q}_{tt'}$  depends on the time difference  $t - t'$  only, i.e., in the energy domain we have  $\check{Q}_{\epsilon, \epsilon'}$

$= 2\pi\delta(\epsilon - \epsilon') \check{Q}(\epsilon)$ . Varying the action with respect to the quantum components  $\phi_2$  and  $\mathbf{a}_2$  and setting them to zero one obtains the Maxwell equations. In the absence of an external magnetic field and/or voltage drops, the mean-field electromagnetic field vanishes,  $\phi = \mathbf{a} = 0$ . In order to have a closed system of equations, one has to supply Eq. (3.2) with the gap equation. Varying the action (2.27) with respect to  $\Delta_2^*$  and using Eq. (3.2) we get the self-consistency equation for the order parameter:

$$\Delta = -\frac{\pi\lambda}{4} \int \frac{d\epsilon}{2\pi} \text{tr}_\Omega Q^K(\epsilon) \tau_-, \quad (3.4)$$

where the four-dimensional space  $\Omega$  is defined in Sec. II.

To clarify the structure of the saddle point given by Eqs. (3.2) and (3.4), consider first the case of nonsuperconducting ( $\Delta = 0$ ) metal. Then it is easy to check that a diagonal in the energy space matrix

$$\Lambda(\epsilon) = \Lambda_0(\epsilon) \Xi, \quad (3.5)$$

where

$$\Lambda_0(\epsilon) = \begin{pmatrix} 1 & 2F(\epsilon) \\ 0 & -1 \end{pmatrix}_K, \quad (3.6)$$

is a solution of Eq. (3.2). The  $4 \times 4$  matrix function  $F(\epsilon)$  introduced in Eq. (3.6) has the meaning of a generalized distribution function. In the steady state it reduces to the single scalar function  $F(\epsilon) = 1 - 2f(\epsilon)$  where  $f(\epsilon)$  is the usual Fermi distribution function. The form of  $f(\epsilon)$  is not fixed by the saddle-point equation since  $\Lambda(\epsilon)$  with arbitrary  $F(\epsilon)$  satisfies Eq. (3.2). The reason for this ambiguity is that any distribution function is allowed in the absence of interaction. To bring the system into the equilibrium with

$$F_{\text{eq}}(\epsilon) = \tanh \frac{\epsilon}{2T}, \quad (3.7)$$

either electron-electron or electron-phonon inelastic interactions must be taken into account. The Keldysh formalism is suitable for the study of nonequilibrium problems as well. In this case there is an externally controlled difference of temperature and/or chemical potential across the system, and the function  $F(\epsilon, \mathbf{r})$  should be obtained from Eq. (3.2) with the proper boundary conditions.

The solution (3.5) captures the eigenvalue structure of a generic saddle point. All fluctuations of  $\check{Q}$  that alter the eigenvalues  $\pm 1$  are massive. The massless modes share the eigenvalue structure of  $\Lambda$  and can be obtained from it by the following transformation:

$$\check{Q} = U^{-1} \Lambda U, \quad (3.8)$$

where  $U$  is some rotation matrix which acts in the  $8 \times 8$  space  $K \otimes \Omega$  as well as in the time domain. According to Eq. (3.8), the field  $\check{Q}$  satisfies the nonlinear constraint

$$\check{Q}^2 = 1$$

at the saddle-point manifold (SPM). Together with Eq. (3.3), this suggests the following parametrization of the Keldysh block:

$$Q^K = Q^R F - F Q^A, \quad (3.9)$$

where, again,  $F$  has the meaning of a generalized distribution function.

Now let us turn to the case of a uniform bulk superconductivity. Here it is convenient to chose a representation of the space  $\Omega$  as a direct product of the Nambu and spin spaces,  $\Omega = N \otimes S$ ; in this notations  $\Xi = \tau_z$ . The superconducting saddle-point solution,  $\check{Q}_S$ , has the form (3.3) with<sup>18</sup>

$$Q_S^{R,A}(\epsilon) = \pm \frac{1}{\sqrt{(\epsilon \pm i0)^2 - |\Delta|^2}} \begin{pmatrix} \epsilon & \Delta \\ -\Delta^* & -\epsilon \end{pmatrix}_N. \quad (3.10)$$

Taken at the saddle-point manifold, the matrix  $\check{Q}$  is equivalent to the Larkin-Ovchinnikov<sup>18</sup> quasiclassical Green function  $\check{g}$ . The mean-field value of the order parameter can be obtained from Eq. (3.4). Substituting  $Q^K$  from Eqs. (3.9) and (3.10) we obtain the standard BCS gap equation (negative  $\lambda$  corresponds to attraction between electrons)

$$\Delta = -\lambda \Delta \int_{|\Delta|}^{\omega_D} \frac{d\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \tanh \frac{\epsilon}{2T}. \quad (3.11)$$

The superconducting saddle point,  $\check{Q}_S$ , belongs to the metallic SPM given by Eq. (3.8). However, in the presence of superconductivity, some excitations on the metallic SPM, having been massless, acquire a gap proportional to  $\Delta$ . A detailed discussion of the hierarchy of gaps in the  $\sigma$  model for N-S systems can be found in Ref. 32.

The next step in the derivation of the  $\sigma$  model is to consider fluctuations of the SPM and to perform the gradient expansion of the action (2.27). Such a procedure is justified in the dirty limit,  $\Delta \tau \ll 1$ , which will be implied from now on. It is equivalent to the replacement of the full Eilenberger<sup>33</sup> equations in the conventional theory of superconductivity to their approximation proposed by Usadel.<sup>21</sup>

As was recently suggested in Ref. 22, in studying electric field fluctuations it is convenient to single out the gauge degrees of freedom in  $\check{Q}$  by the transformation

$$\check{Q}_{t't'} = e^{i\vec{K}(t)\Xi} Q_{t't'} e^{-i\vec{K}(t')\Xi}, \quad (3.12)$$

where  $\vec{K} = K_i \gamma^i$  is related to the doublet  $\vec{K} = (K_1, K_2)^T$ , in analogy with the field  $\vec{\phi}$ . After the transformation (3.12), the action can still be written in the form (2.27), with the Green function being substituted by

$$G^{-1} = i\Xi \frac{\partial}{\partial t} + \frac{(\Xi \nabla - i\vec{\mathbf{A}})^2}{2m} + \frac{i}{2\tau} Q + \vec{\Phi} + \vec{\Delta}_K, \quad (3.13)$$

with

$$\vec{\mathbf{A}} \equiv \vec{\mathbf{a}} - \nabla \vec{\Phi}, \quad (3.14a)$$

$$\vec{\Phi} \equiv \vec{\phi} - \partial_t \vec{K}, \quad (3.14b)$$

$$\vec{\Delta}_K \equiv e^{-i\vec{K}\Xi} \vec{\Delta} e^{i\vec{K}\Xi}. \quad (3.14c)$$

Expanding  $\text{Tr} \ln G^{-1}$  in the standard way,<sup>16,22</sup> we obtain the following effective action:

$$S = S_\sigma + 2\nu \text{Tr} \vec{\Phi}^T \sigma_x \vec{\Phi} + \text{Tr} \frac{\vec{\mathbf{E}}^T \sigma_x \vec{\mathbf{E}} - \vec{\mathbf{H}}^T \sigma_x \vec{\mathbf{H}}}{4\pi e^2} + \frac{2\nu}{\lambda} \text{Tr} \vec{\Delta}^+ \sigma_x \vec{\Delta}, \quad (3.15)$$

where  $S_\sigma$  is the  $\sigma$ -model action for the matrix field  $Q$ ,

$$S_\sigma = \frac{i\pi\nu}{8} \text{Tr} [D(\partial Q)^2 + 4i(i\Xi \partial_t + \vec{\Phi} + \vec{\Delta}_K)Q]. \quad (3.16)$$

Here  $D$  is the diffusion coefficient and  $\partial$  denotes a long covariant derivative,

$$\partial X \equiv \nabla X - i[\Xi \vec{\mathbf{A}}, X]. \quad (3.17)$$

Derivation of the effective action (3.15) that describes interacting disordered normal/superconducting electron liquid is the main result of this section. The model is formulated in terms of the interacting matter field  $Q$  subject to the nonlinear constraint  $Q^2 = 1$ , electromagnetic fields  $\vec{\phi}$ ,  $\vec{\mathbf{a}}$ , and the pairing potential  $\vec{\Delta}$ . At the present stage, the phase  $\vec{K}$  introduced in Eq. (3.12) is left unspecified. It will be expressed in terms of the electromagnetic potentials in Sec. IV C.

In Eq. (3.16),  $Q$  is an  $8 \times 8$  matrix in the  $K \otimes \Omega$  space. In what follows we will assume that all interactions are spin independent. Then  $Q$  is proportional to the unit matrix in the spin space, and the four-dimensional space  $\Omega = N \otimes S$  collapses into the two-dimensional Nambu space. As a result, the theory will be formulated in terms of the  $4 \times 4$  matrices  $Q_{t't'}$  acting in the  $K \otimes N$  space. The corresponding action can be obtained from Eq. (3.16) by taking the trace over the redundant spin space:

$$S_\sigma = \frac{i\pi\nu}{4} \text{Tr} [D(\partial Q)^2 + 4i(i\tau_z \partial_t + \vec{\Phi} + \vec{\Delta}_K)Q], \quad (3.18)$$

where the operator  $\partial$  is given by Eq. (3.17) with  $\Xi = \tau_z$ .

Varying the action Eq. (3.18) with respect to  $Q$  with the constraint  $Q^2 = 1$  yields the equation

$$D \partial(Q \partial Q) + i[i\tau_z \partial_t + \vec{\Phi} + \vec{\Delta}_K, Q] = 0, \quad (3.19)$$

which (for  $\vec{K} = 0$ ) coincides with the dynamical Usadel equation.<sup>18</sup> In the absence of superconductive coupling,  $\vec{\Delta} = 0$ ,  $Q$  is proportional to  $\tau_z$ :  $Q = Q_{KA} \tau_z$ , and our action (3.18) reduces to the Kamenev-Andreev action<sup>22</sup> for the field  $Q_{KA}$ .

## IV. PERTURBATION THEORY

### A. Free metallic diffusons and Cooperons

In this section we will show how a systematic perturbative expansion of the  $\sigma$  model (3.15) can be developed. Keeping in mind further application to N-S devices with relatively weak proximity-induced coupling (Secs. VI and VII), we will consider fluctuations near the *metallic* saddle point (3.5). For this purpose, it is convenient to parametrize the rotation matrix  $U$  in Eq. (3.8) in terms of another matrix  $W$  subject to the linear constraint

$$W\Lambda + \Lambda W = 0. \quad (4.1)$$

Such a parametrization is not unique and a number of them are widely used in literature (see, e.g., Ref. 16).

In the Keldysh formalism, the saddle point (3.3) is not diagonal and, consequently, the solution of Eq. (4.1) for  $W$  explicitly depends on the distribution function  $F(\epsilon)$ . As a result, even for a noninteracting system, intermediate expressions for the Cooperon and diffuson propagators would depend on the particle distribution. To surmount such an unphysical complication, we note that at the saddle point (3.3),  $Q$  can be diagonalized in the Keldysh space by a nonunitary  $2 \times 2$  matrix

$$u = u^{-1} = \begin{pmatrix} 1 & F \\ 0 & -1 \end{pmatrix}_K, \quad (4.2)$$

that separates the distribution function  $F$  from the retarded and advanced blocks which are determined by the spectral properties only:

$$Q = u \begin{pmatrix} Q^R & 0 \\ 0 & Q^A \end{pmatrix} u. \quad (4.3)$$

The function  $F$  in Eq. (4.2) is the stationary fermionic distribution function and we will assume that the system is in thermal equilibrium, so that  $F(\epsilon)$  is given by Eq. (3.7).

The decomposition (4.3) suggests to pass from the initial  $Q$  representation to a new variable  $\mathcal{Q}$  defined as

$$\mathcal{Q} = u Q u. \quad (4.4)$$

In terms of the new variable  $\mathcal{Q}$ , the  $\sigma$ -model action (3.15) acquires the form

$$S_\sigma = \frac{i\pi\nu}{4} \text{Tr} \left[ D(\partial\mathcal{Q})^2 + 4i \left( i\tau_z \frac{\partial}{\partial t} + u\vec{\Phi}u + u\vec{\Delta}_K u \right) \mathcal{Q} \right], \quad (4.5)$$

with the modified definition of the long derivative:

$$\partial X \equiv \nabla X - i[\tau_z u \vec{\mathbf{A}} u, X]. \quad (4.6)$$

Note that the matrix  $u$  couples to the interaction terms only. For the noninteracting case, the distribution function drops from the  $Q$  action.

At the metallic saddle point for the action (4.5),  $\vec{\Phi} = \vec{\mathbf{A}} = \vec{\Delta} = 0$  and  $Q$  is diagonal,  $Q = u\Lambda_0\tau_z u = \sigma_z\tau_z$ . Gapless fluctuations of  $Q$  can then be parametrized as

$$Q = \mathcal{U}^{-1} \sigma_z \tau_z \mathcal{U}, \quad (4.7)$$

where  $\mathcal{U}$  is a unitary matrix, for which we adopt the exponential parametrization,  $\mathcal{U} = e^{\mathcal{W}/2}$ , in terms of the matrix  $\mathcal{W}$  which anticommutes with  $\sigma_z\tau_z$ ,

$$\{\mathcal{W}, \sigma_z \tau_z\} = 0. \quad (4.8)$$

An explicit expression for  $Q$  in terms of  $\mathcal{W}$  reads

$$(a) \quad W_{\epsilon_1\epsilon_2} \begin{array}{c} \xrightarrow{\epsilon_2} \\ \xleftarrow{\epsilon_1} \end{array} W_{\epsilon_2\epsilon_1}$$

$$(b) \quad \Delta \begin{array}{c} \swarrow \\ \searrow \end{array} Q \quad (c) \quad Q \begin{array}{c} \swarrow \times \\ \searrow \times \end{array}$$

FIG. 1. Basic elements of the diagrammatic technique: (a) diffusion-Cooper propagator; (b)  $Q - \Delta$  interaction vertex; (c) effective interaction  $S_\lambda[Q]$  after eliminating the  $\Delta$  field.

$$Q = e^{-\mathcal{W}/2} \sigma_z \tau_z e^{\mathcal{W}/2} = \sigma_z \tau_z (1 + \mathcal{W} + \mathcal{W}^2/2 + \dots). \quad (4.9)$$

The linear matrix constraint (4.8) can be resolved by introducing eight scalar variables,  $w_i$  and  $\bar{w}_i$  with  $i=0,x,y,z$ , as

$$\mathcal{W} = \begin{pmatrix} w_x \tau_x + w_y \tau_y & w_0 + w_z \tau_z \\ \bar{w}_0 + \bar{w}_z \tau_z & \bar{w}_x \tau_x + \bar{w}_y \tau_y \end{pmatrix}_K. \quad (4.10)$$

The diagonal (off-diagonal) in the Nambu space excitations,  $w_i$  and  $\bar{w}_i$  with  $i=0,z$  ( $i=x,y$ ), correspond to diffusion (Cooper) modes. Extracting the quadratic in  $\mathcal{W}$  part from the  $\sigma$ -model action (4.5),

$$iS^{(2)}[\mathcal{W}] = \frac{\pi\nu}{4} \text{Tr}[(Dq^2 - 2i\sigma_z E)\mathcal{W}(\mathbf{q})\mathcal{W}(-\mathbf{q})], \quad (4.11)$$

we obtain the following correlators for diffusons ( $i=0,z$ ):

$$\begin{aligned} & \langle w_i(\mathbf{q}; \epsilon_1, \epsilon_2) \bar{w}_i(-\mathbf{q}; \epsilon_3, \epsilon_4) \rangle \\ &= -\frac{1}{\pi\nu} \frac{(2\pi)^2 \delta(\epsilon_1 - \epsilon_4) \delta(\epsilon_2 - \epsilon_3)}{Dq^2 - i(\epsilon_1 - \epsilon_2)}, \end{aligned} \quad (4.12)$$

and Cooperons ( $i=x,y$ ):

$$\begin{aligned} & \langle w_i(\mathbf{q}; \epsilon_1, \epsilon_2) w_i(-\mathbf{q}; \epsilon_3, \epsilon_4) \rangle \\ &= -\frac{1}{\pi\nu} \frac{(2\pi)^2 \delta(\epsilon_1 - \epsilon_4) \delta(\epsilon_2 - \epsilon_3)}{Dq^2 - i(\epsilon_1 + \epsilon_2)}, \end{aligned}$$

$$\begin{aligned} & \langle \bar{w}_i(\mathbf{q}; \epsilon_1, \epsilon_2) \bar{w}_i(-\mathbf{q}; \epsilon_3, \epsilon_4) \rangle \\ &= -\frac{1}{\pi\nu} \frac{(2\pi)^2 \delta(\epsilon_1 - \epsilon_4) \delta(\epsilon_2 - \epsilon_3)}{Dq^2 + i(\epsilon_1 + \epsilon_2)}. \end{aligned} \quad (4.13)$$

## B. Diagrammatic technique

Expression (4.9) provides a regular way for perturbative expansion near the metallic saddle point,  $Q = \sigma_z \tau_z$ . Its basic

elements are given by the free correlators (4.12) and (4.13) corresponding to soft diffusion and Cooper modes. Since  $\mathcal{W}_{\epsilon\epsilon'}$  is a matrix in the energy space, its correlators are represented diagrammatically by two parallel lines, each of them carrying one energy index, see Fig. 1(a). Expanding the action (4.5) over  $\mathcal{W}$  generates nonlinear vertices describing interaction between diffusion and Cooper modes. The resulting diagrammatics looks very similar to the standard cross technique for dirty metals<sup>34</sup> where soft modes are constructed from two Greens functions averaged over disorder. Note however that in the present case both diffusons and Cooperons are depicted in the same manner, with arrows pointing in the opposite directions on two lines of the propagator. Such a convention, though being unusual for Cooper-

ons, is consistent with the definition of the Fourier-transformed variables (3.1). Diffusons and Cooperons can be distinguished by their structure in the Nambu (and Keldysh) space.

The normalization of the functional integral,  $Z=1$ , manifests itself in cancellation of closed loops in the perturbation theory. In the Keldysh formalism, such a cancellation is related to the integral over the internal energy (let it be  $E$ ) of a closed loop. Indeed, all propagators along the loop have poles in one (upper or lower) half-plane of the complex variable  $E$ . Therefore, integrating over  $E$  yields to cancellation of the corresponding diagram.

For future reference we present here the contraction rule for averaging over  $\mathcal{W}$ :

$$\begin{aligned} \langle \text{Tr } \mathcal{A} \mathcal{W} \cdot \text{Tr } \mathcal{B} \mathcal{W} \rangle = & -\frac{1}{2\pi\nu} \int \frac{d\mathbf{q} d\epsilon_1 d\epsilon_2}{(2\pi)^{d+2}} \left\{ \frac{\text{tr}(AB - A\sigma_z B\sigma_z + A\tau_z B\tau_z - A\sigma_z \tau_z B\sigma_z \tau_z)(Dq^2 + i(\epsilon_1 - \epsilon_2)\sigma_z)}{(Dq^2)^2 + (\epsilon_1 - \epsilon_2)^2} \right. \\ & \left. + \frac{\text{tr}(AB + A\sigma_z B\sigma_z - A\tau_z B\tau_z - A\sigma_z \tau_z B\sigma_z \tau_z)(Dq^2 + i(\epsilon_1 + \epsilon_2)\sigma_z)}{(Dq^2)^2 + (\epsilon_1 + \epsilon_2)^2} \right\}, \end{aligned} \quad (4.14)$$

where  $A = \mathcal{A}(\mathbf{q}, \epsilon_1, \epsilon_2)$  and  $B = \mathcal{B}(-\mathbf{q}, \epsilon_2, \epsilon_1)$ . The first (second) term corresponds to diffusons (Cooperons). For  $Dq^2 \gg \epsilon_1, \epsilon_2$ , this expression can be simplified as

$$\begin{aligned} \langle \text{Tr } \mathcal{A} \mathcal{W} \cdot \text{Tr } \mathcal{B} \mathcal{W} \rangle & = -\frac{1}{\pi\nu} \int \frac{d\mathbf{q} d\epsilon_1 d\epsilon_2}{(2\pi)^{d+2}} \frac{\text{tr}(AB - A\sigma_z \tau_z B\sigma_z \tau_z)}{Dq^2}. \end{aligned} \quad (4.15)$$

Apart from the matter field  $\mathcal{Q}$  (or, equivalently,  $\mathcal{W}$ ), the  $\sigma$ -model action (3.15) with  $S_\sigma$  given by Eq. (3.18) contains the electromagnetic potentials and the field  $\vec{\Delta}$ . The former will be considered in the next subsection, while the latter can be easily eliminated by Gaussian integration. Note that the resulting expression does not depend on the Coulomb phase  $\vec{K}$  entering  $\vec{\Delta}_K$  since it can be ‘‘gauged away’’ by the shift of the phase of the integration variable  $\vec{\Delta}$ . In other words,  $\vec{\Delta}_K$  in Eq. (3.18) can be substituted by  $\vec{\Delta}$  provided that the order parameter field is to be integrated out. The resulting contribution to the action reads

$$\begin{aligned} S_\lambda & = \frac{\pi^2 \nu \lambda}{4} \int d\mathbf{r} dt \sum_{i=x,y} \text{tr } \tau_i \mathcal{Q}_{it} \cdot \text{tr } \tau_i \sigma_x \mathcal{Q}_{it} \\ & = \frac{\pi^2 \nu \lambda}{4} \int d\mathbf{r} dt \text{tr } \sigma_x [Q^2 - (\tau_z Q)^2], \end{aligned} \quad (4.16)$$

and is shown diagrammatically in Fig. 1c. Note that this term is conveniently expressed in the original  $\mathcal{Q}$  representation.

The perturbative expansion near the metallic saddle point is justified in the case of repulsive ( $\lambda > 0$ ) interaction in the Cooper channel or when the proximity-induced superconducting coherence is weak enough. Otherwise, deviation

from the metallic point is large and one should use the solution to the Usadel equation as a starting point for the perturbative analysis (cf. a detailed discussion in Ref. 32).

### C. Electromagnetic field fluctuations

In the previous section we have sketched the basic rules of the diagrammatic perturbation theory in the absence of the Coulomb interaction. Here we will discuss how fluctuations of the electromagnetic field can be incorporated into the formalism. As was shown recently by Kamenev and Andreev,<sup>22</sup> a certain choice of the Coulomb phase  $\vec{K}$  introduced in Eq. (3.12) results in a significant simplification of the theory. Such a choice essentially depends on the position of the non-interacting saddle point on the SPM and is quite different for the metallic (3.5) and superconductive (3.10) saddle points. For the metallic (nonsuperconductive) case, we choose, following Ref. 22,  $\vec{K}$  to be a linear functional of  $\vec{\phi}$  and  $\vec{a}$  and require the vanishing of the term linear both in  $W$  and  $\vec{\Phi}$ ,  $\vec{A}$  in the Usadel equation (3.19), or, equivalently, in the  $\sigma$ -model action (3.18). The resulting equation reads

$$D(\nabla \vec{\Delta} - \Lambda_0 \nabla \vec{\Delta} \Lambda_0) + [\vec{\Phi}, \Lambda_0] = 0, \quad (4.17)$$

where the matrix  $\Lambda_0$  is introduced in Eq. (3.6). Equation (4.17) is to be used to express  $\vec{K}$  in terms of the electromagnetic field potentials  $\vec{\phi}$  and  $\vec{a}$ , the corresponding relation having the form (in this section we share most of notations of Ref. 22)

$$\mathcal{D}^{-1}(\omega, q) \vec{K}(\mathbf{q}, \omega) = \Pi_\omega^{-1} \vec{\phi}(\mathbf{q}, \omega) + iD\sigma_x \mathbf{q} \vec{a}(\mathbf{q}, \omega). \quad (4.18)$$

Here



$$\mathcal{D}^{-1}(\omega, q) = \begin{pmatrix} 0 & Dq^2 + i\omega \\ Dq^2 - i\omega & -2i\omega B_\omega \end{pmatrix}, \quad (4.19)$$

$$\Pi_\omega^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 2B_\omega \end{pmatrix}, \quad (4.20)$$

and

$$B_\omega = \coth \frac{\omega}{2T} \quad (4.21)$$

is the equilibrium bosonic distribution function.

So far our analysis holds for any gauge and any geometry of the sample. From now on we chose the gauge  $\vec{\mathbf{a}}=0$  neglecting relativistic effects due to the magnetic-field fluctuations. Also we restrict ourselves to the consideration of two-dimensional systems. Then one has to integrate out-of-plane degrees of freedom in the electromagnetic field action  $S_{em} = \text{Tr} \vec{\mathbf{E}}^T \sigma_x \vec{\mathbf{E}} / (4\pi e^2)$ . The result depends on the presence or absence of conducting electrodes that screen the long-range Coulomb interaction (for the sake of simplicity, we set the dielectric permeability of the medium to unity). For a single plane, one obtains

$$S_{em} = \int dt \int \frac{d\mathbf{q}}{(2\pi)^2} \text{tr} \vec{\phi}^T(-\mathbf{q}) \sigma_x V_0^{-1}(q) \vec{\phi}(\mathbf{q}), \quad (4.22)$$

where  $V_0(q)$  is the 2D Coulomb interaction potential,

$$V_0(q) = \int \frac{dk_z}{2\pi} \frac{4\pi e^2}{q^2 + k_z^2} = \frac{2\pi e^2}{q}. \quad (4.23)$$

If, for example, there is a metallic gate at a distance  $b$  from the 2D plane, then  $V_0(q)$  is screened in the long-wavelength limit and we have instead

$$V_0^{\text{scr}}(q) = \frac{2\pi e^2}{q} (1 - e^{-2bq}). \quad (4.24)$$

Collecting the terms in the action bilinear in  $\vec{\phi}$  and  $\vec{K}$  and making use of the relation (4.18) we get an effective action for the electromagnetic field propagation in the disordered metal:<sup>22</sup>

$$S_{em}^{\text{eff}} = \text{Tr} \vec{\phi}^T V^{-1} \vec{\phi}, \quad (4.25)$$

where  $V$  has the meaning of a dynamically screened Coulomb interaction in the random-phase approximation,

$$V(q, \omega) = [\sigma_x V_0^{-1}(q) + P_0(q, \omega)]^{-1}, \quad (4.26)$$

where  $P_0(q, \omega)$  is the bare density-density correlator. The matrix  $V(q, \omega)$  has the structure of a bosonic propagator in the Keldysh space:<sup>22</sup>

$$V(q, \omega) = \begin{pmatrix} V^K(q, \omega) & V^R(q, \omega) \\ V^A(q, \omega) & 0 \end{pmatrix}, \quad (4.27)$$

with

$$V^{R,A}(q, \omega) = \left( V_0^{-1}(q) + \frac{2\nu D q^2}{D q^2 \mp i\omega} \right)^{-1}, \quad (4.28a)$$

$$V^K(q, \omega) = B_\omega [V^R(q, \omega) - V^A(q, \omega)]. \quad (4.28b)$$

An additional factor 2 in Eq. (4.28a) compared to that in the Kamenev-Andreev paper<sup>22</sup> is related to the fact that they considered spinless electrons.

Equation (4.25) determines the bare propagator of the electromagnetic field:

$$\langle \phi_i(\mathbf{q}, \omega) \phi_j(-\mathbf{q}, -\omega) \rangle = \frac{i}{2} V_{ij}(q, \omega). \quad (4.29)$$

The propagator of the field  $\vec{K}$  can be obtained from Eq. (4.29) with the help of the relation (4.18) and has the form

$$\langle K_i(\mathbf{q}, \omega) K_j(-\mathbf{q}, -\omega) \rangle = \frac{i}{2} \mathcal{V}_{ij}(q, \omega), \quad (4.30)$$

where the matrix  $\mathcal{V}$  has the same structure as  $V$ , Eq. (4.27), with

$$\mathcal{V}^{R,A}(q, \omega) = - \frac{1}{(D q^2 \mp i\omega)^2} \left( V_0^{-1}(q) + \frac{2\nu D q^2}{D q^2 \mp i\omega} \right)^{-1}, \quad (4.31a)$$

$$\mathcal{V}^K(q, \omega) = B_\omega [\mathcal{V}^R(q, \omega) - \mathcal{V}^A(q, \omega)]. \quad (4.31b)$$

We should note that in Eqs. (4.28a), (4.31a) a possible influence of superconductive pairing upon the dynamic screening of the Coulomb interaction is neglected; this is safe since we will consider N-S systems in the limit of weak tunneling only, so the 2D metal is slightly perturbed by superconductivity.

In a superconductor, the choice of an optimal Coulomb phase  $\vec{K}$  valid in the whole energy range is a complicated task. However, it had been shown in Ref. 35 that in the deep subgap limit ( $\epsilon \ll \Delta$ ) the effect of the electric potential on the quasiclassical Green function  $Q$  is small in the parameter  $\epsilon/\Delta$  and hence  $\vec{K}=0$ . This result will be used below.

## V. RENORMALIZATION OF THE INTERACTION IN THE COOPER CHANNEL

### A. Renormalization-group procedure

In this section we will show how to construct a procedure of successively eliminating high-frequency and high-momentum fluctuations of all interaction modes in the problem: the matrix field  $Q(\mathbf{q}, \epsilon, \epsilon')$ , the order parameter field  $\Delta(\mathbf{q}, \omega)$ , and the electric potential  $\phi(\mathbf{q}, \omega)$ . Elimination of high-energy modes in a dirty 2D metal results in logarithmic corrections to the parameters entering the action and governing dynamics of the retained slow modes. This procedure is known as the renormalization-group (RG) method. We will closely follow Finkelstein's approach to RG construction,<sup>11</sup> with the necessary modifications due to the presence of the Keldysh space instead of replicas.

Each elementary RG step consists in the elimination of degrees of freedom in the energy shell from  $\Omega_*$  to  $\Omega$ , where  $\Omega$  is the current value of the running ultraviolet cutoff in the

problem. Correspondingly, all fluctuating fields are decomposed into fast (denoted by a prime) and slow (denoted by a tilde) parts. For the fields  $\vec{\Delta}$ ,  $\vec{\phi}$ , and  $\vec{K}$  such a representation is trivial:  $\vec{\Delta} = \vec{\Delta} + \vec{\Delta}'$ , etc., while for the field  $Q$  it must be consistent with the constraint  $Q^2 = 1$ . To achieve this, we decompose the rotation matrix  $U$  in Eq. (4.7) into the product of a fast,  $U' = \exp(\mathcal{W}'/2)$ , and a slow,  $\tilde{U}$ , part, so that

$$Q = \tilde{U}^{-1} Q' \tilde{U}, \quad (5.1)$$

where the fast  $Q'$  is expressed in terms of  $\mathcal{W}'$  according to Eq. (4.9).

The fast matrix  $\tilde{\mathcal{W}}'$  is nonzero under the following conditions:

$$\mathcal{W}(\mathbf{q}, \epsilon, \epsilon') \neq 0, \quad \text{if } \Omega_* < \{Dq^2 \text{ or } |\epsilon - \epsilon'|\} < \Omega \quad (5.2)$$

(for diffusions), or

$$\mathcal{W}(\mathbf{q}, \epsilon, \epsilon') \neq 0, \quad \text{if } \Omega_* < \{Dq^2 \text{ or } |\epsilon + \epsilon'|\} < \Omega \quad (5.3)$$

(for Cooperons). The corresponding fast degrees of freedom are excluded from the slow matrix  $\tilde{U}$ . After integration over fast variables,  $\Omega_*$  becomes a new cutoff and the whole procedure should be successively repeated.

In a 2D dirty metal, integrating out fast degrees of freedom results in a relative correction  $\propto \ln(\Omega/\Omega_*) = \zeta_* - \zeta$  to the parameters of the effective action (3.15), where  $\zeta$  is a logarithmic variable defined as

$$\zeta = \ln \frac{1}{\Omega \tau}. \quad (5.4)$$

The RG near the metallic saddle point  $\sigma_z \tau_z$  is justified provided that the Cooper-channel coupling constant  $\lambda \ll 1$  while the dimensionless conductance of the metal  $g \gg 1$ . The latter is defined as

$$g = 2\nu D = \frac{\sigma}{e^2}, \quad (5.5)$$

with  $\sigma = R_{\square}^{-1}$  being the conductance per square [in conventional units,  $\sigma = (e^2/\hbar)g$ ]. Logarithmic corrections to the conductance become large at the localization scale  $\xi \sim g$ . The same is true for the coupling constants  $\Gamma$  and  $\Gamma_2$  omitted in the derivation of the action (3.15), cf. discussion in Sec. II. On the contrary, corrections to  $\lambda$  become of the relative order of unity at much shorter scale, at  $\xi \sim \sqrt{g}$ . Therefore, it is possible, at large  $g$ , to neglect renormalization of the conductance and consider  $g$  as a constant.

### B. BCS correction to $\lambda$

First of all we show how to obtain the standard BCS renormalization of the Cooper-channel interaction constant  $\lambda$  in the present formalism. The correction originates from the term  $S_{\Delta Q} = -\pi\nu \text{Tr} \vec{\Delta} Q$  in the  $\sigma$ -model action (3.18) after eliminating high-frequency fluctuations of the field  $Q$  (here we substitute  $\Delta_K$  by  $\vec{\Delta}$  as was explained in Sec. IV B). Passing to the rotated  $Q$  representation (4.4), expanding to the first power in  $\mathcal{W}'$  according to Eq. (4.9) and setting  $\tilde{U}$  to unity, one obtains the relevant interaction vertex

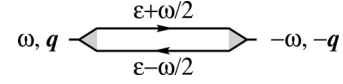


FIG. 2. BCS correction to  $\lambda$ .

$$S_{\text{int}} = -\pi\nu \text{Tr} u \vec{\Delta} u \sigma_z \tau_z \mathcal{W}'. \quad (5.6)$$

After averaging over the fast  $\mathcal{W}'$ , this term will generate the following correction to the action:

$$\Delta S = \frac{i}{2} \langle S_{\text{int}}^2 \rangle. \quad (5.7)$$

The corresponding diagram is shown in Fig. 2. The only fast variable is the internal energy of the  $\langle \mathcal{W}' \mathcal{W}' \rangle$  propagator. Employing then the contraction rule (4.14) and the relation  $\{\vec{\Delta}, \tau_z\} = 0$ , we conclude that the diffuson contribution vanishes identically while the Cooperon pairing yields (to logarithmic accuracy)

$$\Delta S = -\frac{\pi\nu}{2} \int \frac{d\mathbf{q} d\omega d\epsilon}{(2\pi)^4} \frac{1}{\epsilon} \text{tr} \vec{\Delta}(\mathbf{q}, \omega) \vec{\Delta}(-\mathbf{q}, -\omega) \Lambda_0(\epsilon), \quad (5.8)$$

where the  $\epsilon$  integration is performed over the energy shell  $\Omega_* < |\epsilon| < \Omega$ , while  $\mathbf{q}$  and  $\omega$  are restricted to the domain  $Dq^2, |\omega| < \Omega_*$ . On deriving Eq. (5.8) we have used the relation  $\Lambda_0 = u \sigma_z u$ . Now using the definition of the matrices  $\vec{\Delta}$  and  $\Lambda_0$  and omitting the tilde sign over the designation of the slow component of  $\vec{\Delta}$ , we transform the trace in the above equation to the form

$$\begin{aligned} & \text{tr}(\tau_+ \Delta_i - \tau_- \Delta_i^*)(\tau_+ \Delta_j - \tau_- \Delta_j^*) \gamma^i \gamma^j \Lambda_0(\epsilon) \\ & = -4F(\epsilon) (\Delta_1^* \Delta_2 + \Delta_1 \Delta_2^*). \end{aligned} \quad (5.9)$$

As a result, Eq. (5.8) can be represented as

$$\Delta S = 2\nu \text{Tr} \vec{\Delta}^+ \sigma_x \vec{\Delta} \cdot \int_{\Omega_*}^{\Omega} \frac{d\epsilon}{\epsilon} \tanh \frac{\epsilon}{2T}. \quad (5.10)$$

The temperature  $T$  thus determines the infrared cutoff of the RG procedure. At larger scales,  $\Omega \gg T$ , one has

$$\Delta S = 2\nu \ln \frac{\Omega}{\Omega_*} \text{Tr} \vec{\Delta}^+ \sigma_x \vec{\Delta}. \quad (5.11)$$

Comparing with the last term in the action (3.15), we conclude that the correction (5.11) renormalizes  $1/\lambda$ :

$$\frac{\partial(1/\lambda)}{\partial \zeta} = 1. \quad (5.12)$$

For a superfluid Fermi liquid, the RG equation (5.12) was derived in Ref. 36; in that case only the Cooper channel was relevant, therefore the RG approach was equivalent to a simple summation of the standard BCS-theory ladder.

Coulomb interaction in dirty superconductors contributes both to the Cooper and to the density-density channels. We consider here the range of parameters where  $\ln(1/\Omega \tau) \ll g$  that makes it possible to neglect the effect of the Cooper channel upon the conductance  $g$ ; however, the effect of the density-density channel upon the Cooper one has the relative order of  $g^{-1} \ln^2(1/\Omega \tau)$  and thus should be taken into ac-

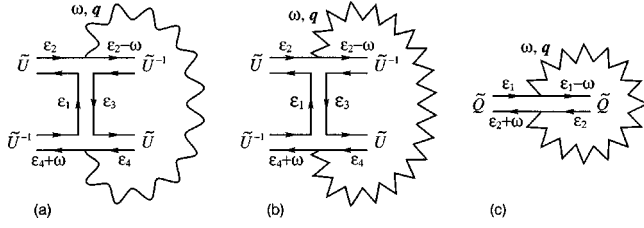


FIG. 3. Diagrams for the Coulomb correction to  $\lambda$ : (a)  $\langle (S_{\text{int}}^a)^2 \rangle$ , wavy line denotes the correlator  $\langle \phi_i(\mathbf{q}, \omega) \phi_j(-\mathbf{q}, -\omega) \rangle$ ; (b)  $\langle (S_{\text{int}}^b)^2 \rangle$ , zigzag line stands for the correlator  $\langle K_i(\mathbf{q}, \omega) K_j(-\mathbf{q}, -\omega) \rangle$ ; (c)  $\langle S_{\text{int}}^c \rangle$ .

count. This effect can be described in the form of an integral equation for the energy-dependent Cooper attraction  $\lambda(E)$ , as was done by Aleiner and Altshuler<sup>37</sup> (the same kind of equation was derived in Ref. 38 for another but similar problem). For our purpose it will be more convenient to treat the same effect within the RG procedure, as described in the next subsection.

### C. Coulomb correction to the Cooper-channel interaction

In this subsection we calculate the Coulomb-induced correction to the coupling constant  $\lambda$ . It appears as a result of eliminating high-momentum fluctuations of the electric field. According to Eq. (4.5), the electric field couples to the matter field  $Q$  by the following terms:

$$\begin{aligned} S_{\text{int}} &= -\pi\nu \text{Tr} \left[ u(\vec{\phi} - \partial_t \vec{K}) u Q + \frac{1}{2} D \tau_z u \nabla \vec{K} u [Q, \nabla Q] \right. \\ &\quad \left. + \frac{i}{4} D [\tau_z u \nabla \vec{K} u, Q] \right]^2 \\ &\equiv S_{\text{int}}^a + S_{\text{int}}^b + S_{\text{int}}^c. \end{aligned} \quad (5.13)$$

Here we utilize the relation (4.18) connecting the phase  $\vec{K}$  to the field  $\vec{\phi}$ . The interaction vertices  $S_{\text{int}}^a$  and  $S_{\text{int}}^b$  are linear in  $\vec{\phi}$ , while the vertex  $S_{\text{int}}^c$  is quadratic. Then the result of averaging over fast variables can be written as

$$\Delta S = \frac{i}{2} \langle [S_{\text{int}}^a + S_{\text{int}}^b]^2 \rangle + \langle S_{\text{int}}^c \rangle. \quad (5.14)$$

Consider first the average  $\langle (S_{\text{int}}^a)^2 \rangle$ . The relevant (i.e., logarithmic) contribution can be written in the form

$$\langle (S_{\text{int}}^a)^2 \rangle = \pi^2 \nu^2 \langle [\text{Tr} \tilde{U} u(\vec{\phi}' - \partial_t \vec{K}') u \tilde{U}^{-1} \sigma_z \tau_z \mathcal{W}']^2 \rangle, \quad (5.15)$$

where the pairings are shown in Fig. 3(a). According to this diagram, all energies are coupled to slow variables, and there is only one fast variable, the internal momentum  $\mathbf{q}$  running over  $\langle \phi' \phi' \rangle$  and  $\langle \mathcal{W}' \mathcal{W}' \rangle$ . Therefore, to logarithmic accuracy we may consider  $Dq^2 \gg \omega$ . In this limit, one can neglect the term  $i\omega \vec{K}$  compared to  $\vec{\phi}$  as follows from Eq. (4.18) and use the universal large- $q$  asymptotics of the screened Coulomb interaction (4.27),  $\langle \phi_i(\mathbf{q}, \omega) \phi_j(-\mathbf{q}, -\omega) \rangle \simeq (i/4\nu) \sigma_x$ , which is independent of the details of the bare potential  $V_0(q)$ . The averaging over  $\mathcal{W}'$  can be performed

with the help of Eq. (4.15), making use of the inequality  $Dq^2 \gg \epsilon_1, \epsilon_3$ . As a result, one obtains

$$\begin{aligned} \langle (S_{\text{int}}^a)^2 \rangle &= -\frac{i\pi}{4} \int \frac{d\mathbf{r} d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 d\omega d\mathbf{q}}{(2\pi)^7} \frac{1}{Dq^2} \\ &\quad \times \sum_{i \neq j} \text{tr}(A_i B_j - A_i \sigma_z \tau_z B_j \sigma_z \tau_z), \end{aligned} \quad (5.16)$$

where the  $\mathbf{q}$  integration is taken over the fast energy shell  $\Omega_* < Dq^2 < \Omega$ , the matrices  $A_i$  and  $B_j$  are given by

$$A_i = \tilde{U}_{\epsilon_1, \epsilon_2} u_{\epsilon_2} \gamma^i u_{\epsilon_2 - \omega} \tilde{U}_{\epsilon_2 - \omega, \epsilon_3}^{-1} \sigma_z \tau_z, \quad (5.17a)$$

$$B_j = \tilde{U}_{\epsilon_3, \epsilon_4} u_{\epsilon_4} \gamma^j u_{\epsilon_4 + \omega} \tilde{U}_{\epsilon_4 + \omega, \epsilon_1}^{-1} \sigma_z \tau_z, \quad (5.17b)$$

and all slow matrices  $U$  are taken at a coincident spacial point  $\mathbf{r}$ . Performing integration over  $\epsilon_1$  and  $\epsilon_3$  in the first term under the trace in Eq. (5.16), we obtain

$$\int \frac{d\epsilon_1 d\epsilon_3}{(2\pi)^2} \sum_{i \neq j} \text{tr} A_i B_j = 2 \text{tr} \sigma_x (u \tilde{Q} u)_{\epsilon_2 - \omega, \epsilon_4} (u \tilde{Q} u)_{\epsilon_4 + \omega, \epsilon_2}, \quad (5.18)$$

where a slow  $\tilde{Q}$  is defined as  $\tilde{Q} = \tilde{U}^{-1} \sigma_z \tau_z \tilde{U}$ . In the second term under the trace in Eq. (5.16), the matrices  $\sigma_z \tau_z$  cancel with those in Eq. (5.17), integration over  $\epsilon_1$  and  $\epsilon_3$  is equivalent to multiplication of  $U^{-1}$  and  $U$  that gives the unit matrix, and tracing with  $\sigma_x$  yields zero.

The resulting expression takes a simple form in the initial  $Q$  representation in the time domain. Integrating over  $\mathbf{q}$ , one obtains (omitting the tilde sign)

$$\langle (S_{\text{int}}^a)^2 \rangle = -\frac{i\nu}{4g} \ln \frac{\Omega}{\Omega_*} \int d\mathbf{r} dt \text{tr} \sigma_x Q_{it}^2(\mathbf{r}). \quad (5.19)$$

The correction to the action  $(i/2) \langle (S_{\text{int}}^a)^2 \rangle$  is the only one in the standard gauge with  $\vec{K} = 0$ . Physical quantities should not depend on the choice of  $\vec{K}$ , which is a kind of a gauge transformation upon the matrix  $Q$  [cf. Eq. (3.12)]. For this reason, it would be enough to use Eq. (5.19) to find the renormalization of the coupling constants in the effective action. Below we present, however, the calculation of other terms in Eq. (5.14) related to  $\vec{K}$ , in order to show that they really cancel each other.

The diagram for

$$\langle (S_{\text{int}}^b)^2 \rangle = \pi^2 \nu^2 D^2 \langle [\text{Tr} \tilde{U} \tau_z u \nabla \vec{K}' u \tilde{U}^{-1} \nabla \mathcal{W}']^2 \rangle \quad (5.20)$$

is shown in Fig. 3(b). It looks similar to the one in Fig. 3(a), with the wavy line being replaced by the zigzag line denoting  $\langle K' K' \rangle$  correlator. Repeating the steps that lead to Eq. (5.16) and using the large- $q$  asymptotics of Eq. (4.31) we obtain

$$\begin{aligned} \langle (S_{\text{int}}^b)^2 \rangle &= \frac{i\pi}{4} \int \frac{d\mathbf{r} d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 d\omega d\mathbf{q}}{(2\pi)^7} \frac{1}{Dq^2} \\ &\quad \times \sum_{i \neq j} \text{tr}(M_i N_j - M_i \sigma_z \tau_z N_j \sigma_z \tau_z), \end{aligned} \quad (5.21)$$

with

$$M_i = \tilde{U}_{\epsilon_1 \epsilon_2} \tau_z u_{\epsilon_2} \gamma^i u_{\epsilon_2 - \omega} \tilde{U}_{\epsilon_2 - \omega, \epsilon_3}^{-1},$$

$$N_j = \tilde{U}_{\epsilon_3, \epsilon_4} \tau_z u_{\epsilon_4} \gamma^j u_{\epsilon_4 + \omega} \tilde{U}_{\epsilon_4 + \omega, \epsilon_1}^{-1}.$$

Integrating over  $\mathbf{q}$ ,  $\epsilon_1$  and  $\epsilon_3$  as described above, we get

$$\langle (S_{\text{int}}^b)^2 \rangle = -\frac{i\nu}{4g} \ln \frac{\Omega}{\Omega_*} \int d\mathbf{r} dt \text{tr} \sigma_x (\tau_z Q_{tt}(\mathbf{r}))^2. \quad (5.22)$$

In the same manner it can be shown that the average  $\langle S_{\text{int}}^a S_{\text{int}}^b \rangle = 0$  vanishes.

In calculating the average  $\langle S_{\text{int}}^c \rangle$  shown diagrammatically in Fig. 3(c), all  $Q$ 's may be considered slow. The analytical expression reads

$$\langle S_{\text{int}}^c \rangle = -\frac{i\pi\nu D}{2} \langle \text{Tr}(\tau_z \bar{Q} \vec{\nabla} \vec{K}')^2 \rangle, \quad (5.23)$$

where the fast momentum runs over the  $\langle K'K' \rangle$  propagator. Calculating the corresponding logarithmic integral, one obtains

$$\langle S_{\text{int}}^c \rangle = -\frac{\nu}{8g} \ln \frac{\Omega}{\Omega_*} \int d\mathbf{r} dt \text{tr} \sigma_x (\tau_z Q_{tt}(\mathbf{r}))^2. \quad (5.24)$$

Substituting Eqs. (5.19), (5.22), and (5.24) into Eq. (5.14), we see that the contributions from the vertices describing interaction with the field  $\vec{K}$  cancel each other, and the resulting expression is given by

$$\begin{aligned} \Delta S &= \frac{\nu}{8g} \ln \frac{\Omega}{\Omega_*} \int d\mathbf{r} dt \text{tr} \sigma_x Q_{tt}^2(\mathbf{r}) \\ &= \frac{\nu}{16g} \ln \frac{\Omega}{\Omega_*} \int d\mathbf{r} dt \sum_{j=0}^3 \text{tr} \sigma_x \tau_j Q_{tt}(\mathbf{r}) \cdot \tau_j Q_{tt}(\mathbf{r}). \end{aligned} \quad (5.25)$$

Comparing with Eq. (4.16), we see that the terms with  $j = x, y$  renormalize the Cooper channel coupling  $\lambda$ , whereas the terms with  $j = 0, z$  contribute to the couplings in the diffusion channel. The latter,  $\Gamma$  and  $\Gamma_2$ , are not taken into account since corrections to them are of the relative order of  $g^{-1} \ln(1/\Omega\tau)$ . As a result, we get the Coulomb contribution to the RG equation for  $\lambda$ :

$$\frac{\partial \lambda}{\partial \zeta} = \frac{1}{4\pi^2 g}, \quad (5.26)$$

which coincides with Finkelstein's result<sup>11,12</sup> in the limit  $\Gamma$ ,  $\Gamma_2 \rightarrow 0$ .

#### D. Solution of the RG equation for $\lambda$ and shift of $T_c$

Combining Eqs. (5.12) and (5.26), we arrive at the complete renormalization-group equation for the Cooper-channel interaction constant:

$$\frac{\partial \lambda}{\partial \zeta} = -\lambda^2 + \frac{1}{4\pi^2 g}. \quad (5.27)$$

In the high-energy range  $1/\tau \ll E \ll E_F$  the second term of Eq. (5.27) is absent. We will use the solution of Eq. (5.12) at  $E \approx \tau^{-1}$  as an initial condition for the full equation (5.27).

The renormalization-group equation (5.27) possesses two fixed points,  $\pm \lambda_g$ , where

$$\lambda_g = \frac{1}{2\pi\sqrt{g}}. \quad (5.28)$$

The stable fixed point,  $+\lambda_g$ , is the limiting point of the RG flow in the metallic region. A trajectory reaches its asymptotic value  $\lambda_g$  at the scale  $\zeta \sim \sqrt{g}$ . The unstable fixed point,  $-\lambda_g$ , separates the regions of metallic ( $\lambda > -\lambda_g$ ) and superconducting ( $\lambda < -\lambda_g$ ) states. The solution of Eq. (5.27) is given by

$$\lambda(\zeta) = \frac{\lambda_0 + \lambda_g \tanh \lambda_g \zeta}{1 + \frac{\lambda_0}{\lambda_g} \tanh \lambda_g \zeta}, \quad (5.29)$$

where  $\lambda_0$  is the bare value of the interaction constant in the Cooper channel defined at the energy scale  $\tau^{-1}$ .

To study the superconductor-metal transition we will consider here the case of an attractive interaction,  $\lambda_0 < 0$ . The superconducting transition temperature  $T_c$  is determined by the position of the pole in  $\lambda(\zeta)$ , with  $\zeta = \ln(1/T_c\tau)$ . In the clean system ( $g \rightarrow \infty$ ),

$$T_{c0}\tau = \exp\left(-\frac{1}{|\lambda_0|}\right). \quad (5.30)$$

For a finite  $\lambda_g$ , the critical value of  $\zeta$  can be easily obtained from Eq. (5.29) and is given by

$$\zeta_c = \frac{1}{2\lambda_g} \ln \frac{|\lambda_0| + \lambda_g}{|\lambda_0| - \lambda_g}. \quad (5.31)$$

Consequently, we get for  $T_c$ :

$$T_c\tau = \left( \frac{|\lambda_0| - \lambda_g}{|\lambda_0| + \lambda_g} \right)^{1/2\lambda_g}.$$

Substitution of  $\lambda$  in terms of  $T_{c0}$  with the help of Eq. (5.30) leads to the final result for  $T_c$  suppression by disorder, which coincides (within our accuracy) with that of Ref. 11:

$$T_c\tau = \left( \frac{1 - \frac{1}{2\pi\sqrt{g}} \ln \frac{1}{T_{c0}\tau}}{1 + \frac{1}{2\pi\sqrt{g}} \ln \frac{1}{T_{c0}\tau}} \right)^{\pi\sqrt{g}}. \quad (5.32)$$

Evaluating expression (5.32) for  $g \gg 1$ , we obtain a perturbative reduction of the transition temperature:<sup>9,10</sup>

$$\ln \frac{T_c}{T_{c0}} = -\frac{1}{12\pi^2 g} \ln^3 \frac{1}{T_{c0}\tau}, \quad (5.33)$$

valid at large conductances. The critical temperature becomes zero and the superconducting transition disappears at the critical value of the dimensionless conductance

$$g_c = \left( \frac{1}{2\pi} \ln \frac{1}{T_c \tau} \right)^2. \quad (5.34)$$

Note once again, that the result (5.32) for  $T_c(g)$  dependence is obtained neglecting weak-localization corrections to the conductance  $g$ , as well as thermal, quantum, and mesoscopic fluctuations. This is correct provided that the renormalized conductance at the scale  $T_c$  is still greater than unity:  $g_* = g - (a/\pi^2) \ln(1/T_c \tau) \gg 1$ , where the constant  $a$  is given by the sum of the usual weak localization and interaction corrections, and is equal either to 1 or 1/4, depending on the absence or presence of the spin-orbit interaction.

In the above derivation we neglected spin-dependent interactions ( $\Gamma_2$  in the notations of Ref. 12) which may change the numerical coefficient in Eq. (5.26). On the other hand, it was explained in Ref. 12 that strong spin-orbit scattering eliminates possible effect of  $\Gamma_2$  upon  $T_c(g)$  dependence.

## VI. ANDREEV CONDUCTANCE

### A. Tunneling term in the action

In the previous sections, we considered a uniform 2D system. In principle, spacial inhomogeneities in the local system's characteristics such as the conductance and/or the Cooper channel interaction  $\lambda$  can be easily incorporated into the  $\sigma$ -model action (3.15) by a spacial dependence of the parameters of the  $\sigma$  model. Thus, the action (3.15) is suitable for a description of N-S interfaces or interfaces between metals with different conductances. However since the solutions of the Usadel equation (3.19) are continuous, only interfaces with perfect transmission  $T=1$  can be described in such a manner. In order to be able to deal with the interfaces of arbitrary transparencies, one has to introduce a boundary term into the action. Below in this paper we consider the case of low-transparent interfaces which can be described by means of the tunneling Hamiltonian approximation. Then the boundary term in the action can be derived in the second order over the tunneling Hamiltonian and reads

$$S_\gamma = -\frac{i\pi}{4} \gamma \text{Tr}_\Gamma \check{Q}^{(1)} \check{Q}^{(2)}. \quad (6.1)$$

Here  $\check{Q}^{(1)}$  and  $\check{Q}^{(2)}$  refer to different sides of the interface boundary  $\Gamma$ ; the notation  $\text{Tr}_\Gamma$  means that the space integral is taken over the interface surface, and  $\gamma$  is the (dimensionless) normal-state tunneling conductance per unit area of the boundary. In the expression (6.1) the trace over the spin space has already been performed.

Variation of the total action  $S_{\text{tot}} = S_\sigma + S_\gamma$ , with  $S_\sigma$  given by Eq. (3.18), with respect to  $\check{Q}$ , reproduces the matrix Usadel equations for  $\check{Q}^{(1)}$  and  $\check{Q}^{(2)}$  together with the corresponding boundary conditions<sup>39,32</sup> (cf. similar derivation in Ref. 40):

$$g_1 \check{Q}^{(1)} \nabla_\perp \check{Q}^{(1)} = g_2 \check{Q}^{(2)} \nabla_\perp \check{Q}^{(2)} = \frac{\gamma}{2} [\check{Q}^{(1)}, \check{Q}^{(2)}], \quad (6.2)$$

where  $\nabla_\perp$  stands for the gradient along the normal to the interface directed from the medium (1) to the medium (2) [Eq. (6.2) is written in the form assuming the absence of magnetic field]. It amounts to a straightforward calculation to

show that the action (6.1) leads to the following expression for the bare normal-state tunneling conductance  $\sigma_T$ :

$$\sigma_T = \frac{e^2}{\hbar} \mathcal{A} \gamma, \quad (6.3)$$

where  $\mathcal{A}$  is the area of the tunnel junction. We omit here such a calculation since it is fairly similar to the calculation of the Andreev subgap conductance presented in the next subsection. Similarly, we will not dwell upon Coulomb interaction-induced corrections to the tunneling conductance,<sup>7,41</sup> which can be derived from the action (6.1) by taking into account fluctuations of the fields  $Q$ ,  $\phi$ , and  $K$  (cf. Ref. 22). An analogous calculation of the interaction effects in the Andreev conductance is one of our main subjects below.

### B. Andreev conductance in the effective action formalism

In this subsection we rederive, within the effective action formalism, the well-known results for the subgap Andreev conductance  $G_A$  between a superconductor and a dirty normal metal (cf., e.g. Ref. 42). We start from the simplest situation when  $G_A$  does not depend on voltage and/or frequency. First of all we show that if the effective action contains the following term:

$$S_A = \frac{i\pi}{16} G_A \text{Tr}(Q_S \Lambda)^2, \quad (6.4)$$

then  $G_A$  is indeed the dimensionless subgap conductance. In the second step, we prove that the term of the form (6.4) is generated in the second order of expansion over  $S_\gamma$ .

To begin with, we note that at low energies,  $\epsilon \ll \Delta$ , the superconductive matrix Greens function  $Q_S$  does not depend on  $\epsilon$  and reduces purely to phase rotations:

$$Q_S = \frac{\vec{\Delta}}{i|\Delta|}, \quad (6.5)$$

where  $\vec{\Delta}$  can be obtained from Eq. (2.25). Therefore, we can perform the trace in Eq. (6.4) over energy variables and the Keldysh matrix space, using Eq. (6.5) and the formula<sup>22</sup>

$$\begin{aligned} \int dE \text{tr}_K [\gamma^i \gamma^j - \gamma^i \Lambda_0(E + \omega/2) \gamma^j \Lambda_0(E - \omega/2)] \\ = 4\omega (\Pi_\omega^{-1})^{ij}, \end{aligned} \quad (6.6)$$

where the matrix  $\Pi_\omega^{-1}$  is defined in Eq. (4.20). As a result, we obtain the effective action as a functional of the order parameter:

$$S_A = -\frac{iG_A}{8|\Delta|^2} \text{tr}_N \int \frac{d\omega}{2\pi} \vec{\Delta}^T(-\omega) \omega \Pi_\omega^{-1} \vec{\Delta}(\omega), \quad (6.7)$$

where  $\Delta_i = 1/2 \text{tr}_K \vec{\Delta} \gamma^i$ . On deriving this equation we added the constant term  $Q_S^2 = 1$  under the trace in Eq. (6.4). Employing Eq. (2.25), one can rewrite Eq. (6.7) in terms of the phase variables  $\theta$  as

$$\begin{aligned}
S_A[\theta_i] = & \frac{iG_A}{4} \int \frac{d\omega}{2\pi} \left\{ i\omega [(e^{i\theta_1} \cos \theta_2)_{-\omega} (e^{-i\theta_1} \sin \theta_2)_{\omega} \right. \\
& - (e^{-i\theta_1} \cos \theta_2)_{-\omega} (e^{i\theta_1} \sin \theta_2)_{\omega}] \\
& \left. + 2\omega \coth \frac{\omega}{2T} (e^{i\theta_1} \sin \theta_2)_{-\omega} (e^{-i\theta_1} \sin \theta_2)_{\omega} \right\}.
\end{aligned} \tag{6.8}$$

Expression (6.8) for the action makes it possible to relate the coefficient  $G_A$  in Eq. (6.4) with the Andreev conductance of the N-S interface. For this purpose let us suppose that the superconducting island is biased at some voltage  $V(t)$ . Then the phase of the island will rotate with the speed  $2eV$ . In the Keldysh formalism this corresponds to the rotation of its classical component,

$$\frac{d\theta_1}{dt} = 2eV. \tag{6.9}$$

To find the Andreev current,  $I = 2e(dn/dt)$ , where  $n$  is the number of the Cooper pairs on the island, one may use the fact that  $n$  and  $\theta$  are canonically conjugated variables, and thus,  $\hat{I} = 2ie(\delta/\delta\theta)$ . Translating this into the Keldysh formalism, we have, similar to Ref. 22,

$$\langle I(t) \rangle = ie \left. \frac{\delta \ln Z}{\delta \theta_2(t)} \right|_{\theta_2=0} = ie \left. \frac{\delta Z}{\delta \theta_2(t)} \right|_{\theta_2=0}, \tag{6.10}$$

where the last equation follows from the normalization condition  $Z=1$  in the absence of quantum sources.

Transforming Eq. (6.8) with the help of Eq. (6.9), we get

$$S_A = -eG_A \int V(t) \theta_2(t) dt + o[\theta_2]. \tag{6.11}$$

Substituting this action into Eq. (6.10) and taking the functional derivative, we obtain  $\langle I(t) \rangle = e^2 G_A V(t)$ . Hence, the Andreev conductance of the N-S interface in conventional units is equal to

$$\sigma_A = \frac{e^2}{\hbar} G_A. \tag{6.12}$$

This completes the proof of the physical meaning of the term  $S_A$ , Eq. (6.4), in the action. An alternative proof is presented in Appendix A, where we calculate the voltage noise at the N-S barrier (related to the conductance  $\sigma_A$  due to the fluctuation-dissipation relation). Now we turn to the derivation of such a term in the simplest geometry of N-S contact.

### C. Rectangular N-S contact

#### 1. Semiclassical solution within the effective action method

The simplest example of N-S contact is shown in Fig. 4. Superconductor (S) is connected to a normal reservoir (R) via a thin film of dirty metal (N) of length  $L_x$  and width  $L_y$ . The relevant energy scale in the N region is determined by the Thouless energy  $E_{Th} = D/L_x^2$ . As long as temperature, voltage, and frequency of measurement are all small enough,  $\max(eV, T, \omega) \ll E_{Th}$ , the Andreev conductance is just a constant:  $\sigma_A = \sigma_T^2 R_D$ , where  $R_D = \sigma^{-1} L_x/L_y$  is the resistance of

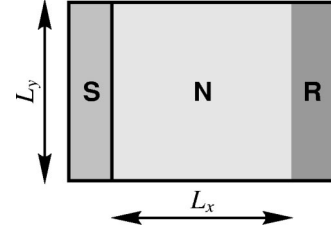


FIG. 4. Rectangular N-S contact. Superconductor (S) is connected to a normal reservoir (R) via a dirty metal film (N) of length  $L_x$  and width  $L_y$ .

the N region<sup>2</sup>. Below we first show how to get such a result [described by the term (6.4) in the action] within the effective action method, and later on we turn to its generalizations, taking into account finite-energy effects as well as effects due to the interaction in the Cooper channel and due to the zero-bias anomaly.

Consider the boundary action  $S_\gamma$  defined in Eq. (6.1) in the case when  $Q^{(1)} = Q_S$ ,  $Q^{(2)} = Q$ . In the present geometry it is convenient to treat the boundary as a *line* between N and S films, then  $\gamma$  becomes the tunneling conductance *per unit length*. To obtain the Andreev term  $S_A$ , we need to expand the functional integral for  $Z$  up to the second order over  $S_\gamma$  and average over fluctuations of the normal-metal matrix  $Q$ , i.e., over Cooperon modes:

$$S_A = \frac{i}{2} \langle S_\gamma^2 \rangle = -\frac{i\pi^2 \gamma^2}{32} \left\langle \left( \int_0^{L_y} dy \text{tr} Q_S Q(0, y) \right)^2 \right\rangle \tag{6.13}$$

$$\begin{aligned}
&= -\frac{i\pi^2 \gamma^2}{32} \int_0^{L_y} dy_1 \int_0^{L_y} dy_2 \\
&\quad \times \langle \text{tr} Q_S \sigma_z \tau_z \mathcal{W}(0, y_1) \cdot \text{tr} Q_S \sigma_z \tau_z \mathcal{W}(0, y_2) \rangle
\end{aligned} \tag{6.14}$$

Proceeding from Eq. (6.13) to Eq. (6.14) we switched to the ‘‘rotated’’ representation of the  $Q$  matrices defined in Eq. (4.4) and used the representation of  $Q$  in terms of the generators  $\mathcal{W}$  as defined in Eq. (4.9). Note that we neglect here any possible fluctuations of the superconductive matrix field  $Q_S$ . The next step is to expand the field  $\mathcal{W}(x, y)$  over the eigenfunctions  $\psi_{mn}(x, y)$  of the diffusion equation:

$$\mathcal{W}(x, y) = \sum_{m, n} \mathcal{W}_{mn} \psi_{mn}(x, y). \tag{6.15}$$

The choice of the eigenfunctions is determined by the boundary conditions, which are vanishing of current at the edges  $y=0$ ,  $y=L_y$  and at the N-S boundary  $x=0$  (the last condition is an approximation valid due to smallness of  $G_T R_D$ ), and vanishing of the Cooperon amplitude at the boundary with the normal reservoir. These conditions result in the following set of eigenfunctions:

$$\begin{aligned}
\psi_{mn}(x, y) &= \frac{2 \cos q_m x \cos k_n y}{\sqrt{(1 + \delta_{n,0}) L_x L_y}}, \quad q_m = \frac{\pi}{L_x} \left( m + \frac{1}{2} \right), \\
k_n &= \frac{\pi}{L_y} n, \quad m, n = 0, 1, \dots
\end{aligned} \tag{6.16}$$

After the integration over  $y$  in Eq. (6.14), only zero mode  $n=0$  survives. Using Eq. (4.14) for the contraction rule and the fact that  $\{Q_S, \tau_z\}=0$ , we conclude that the diffusion pairing in Eq. (4.15) does not contribute, whereas the Cooperon one does, so we get

$$S_A = \frac{i\pi(\gamma L_y)^2}{16\nu L_x L_y} \int \frac{d\epsilon_1 d\epsilon_2}{(2\pi)^2} \sum_m \frac{\text{tr}[Q_S(\epsilon_1, \epsilon_2)\sigma_z \tau_z Q_S(\epsilon_2, \epsilon_1)\sigma_z \tau_z - Q_S(\epsilon_1, \epsilon_2)Q_S(\epsilon_2, \epsilon_1)][Dq_m^2 + i(\epsilon_1 + \epsilon_2)\sigma_z]}{(Dq_m^2)^2 + (\epsilon_1 + \epsilon_2)^2}$$

$$= \frac{i\pi(\gamma L_y)^2}{16\nu L_x L_y} \int \frac{dE d\omega}{(2\pi)^2} \sum_m \frac{\text{tr}[Q_S(\omega)\Lambda(E_-)Q_S(-\omega)\Lambda(E_+) - Q_S(\omega)Q_S(-\omega)][Dq_m^2 + 2iE\Lambda_0(E_+)]}{(Dq_m^2)^2 + 4E^2}, \quad (6.17)$$

where  $E_{\pm} = E \pm \omega/2$ . Using the property  $Q_S^2 = 1$  one can verify that the contribution of the term with  $iE\Lambda_0(E_+)$  in the numerator behaves as  $\omega^2$  for small  $\omega$ . For this reason it can be neglected compared to the  $\omega$ -proportional kernel in Eq. (6.7). However, we will see below that the term of this structure determines the amplitude of the Josephson proximity coupling. Then, the term with  $Dq_m^2 Q_S(\omega)Q_S(-\omega)$ , being integrated over  $\omega$ , produces an unimportant constant that can be omitted. As a result, one has

$$S_A = \frac{i\pi(\gamma L_y)^2}{16\nu L_x L_y} \int \frac{dE d\omega}{(2\pi)^2} \times \sum_m \frac{Dq_m^2 \text{tr}[Q_S(\omega)\Lambda(E_-)Q_S(-\omega)\Lambda(E_+)]}{(Dq_m^2)^2 + 4E^2}. \quad (6.18)$$

In the limit  $E \ll E_{Th} = D/L_x^2$  the sum over  $m$  can be easily calculated:  $\sum_m 1/q_m^2 = L_x^2/2$ . The region of energies  $E$  which are relevant for the integral in Eq. (6.17) is given by  $E \sim \max(\omega, T)$ . Thus, as long as the condition  $\max(\omega, T) \ll E_{Th}$  is fulfilled, we convert Eq. (6.17) into the foreseen expression of the form (6.4):

$$S = \frac{i\pi}{16} \frac{G_T^2}{g} \frac{L_x}{L_y} \text{Tr}(Q_S \Lambda)^2, \quad (6.19)$$

where  $G_T = \gamma L_y$  is the total normal-state conductance of the interface, according to Eq. (6.3). Now, comparing with Eq. (6.4), one finds for the dimensionless Andreev conductance

$$G_A^{(0)} = \frac{G_T^2}{g} \frac{L_x}{L_y}. \quad (6.20)$$

The above relation coincides with the known<sup>2</sup> relation  $\sigma_A = \sigma_T^2 R_D$ , since in the present geometry  $R_D = \sigma^{-1} L_x / L_y$ .

At higher temperatures or frequencies the  $E$  dependence of the denominator in Eq. (6.17) becomes important and the simple invariant representation (6.4) is not valid anymore. However, one can still present the Andreev term in the action as a slight modification of Eq. (6.7):

$$S_A = - \frac{i}{8|\Delta|^2} \text{tr} \int \frac{d\omega}{2\pi} G_A(\omega) \vec{\Delta}^T(-\omega) \omega \Pi_{\omega}^{-1} \vec{\Delta}(\omega). \quad (6.21)$$

The function  $G_A(\omega)$  is given by

$$G_A^{(0)}(\omega) = \frac{G_T^2}{g} \frac{L_{\text{eff}}(L_x, \omega, T)}{L_y}, \quad (6.22)$$

where

$$L_{\text{eff}}(L_x, \omega, T) = \frac{2D}{L_x \omega} \int_0^{\infty} dE \left[ \tanh \frac{E_+}{2T} - \tanh \frac{E_-}{2T} \right] \times \sum_m \frac{Dq_m^2}{(Dq_m^2)^2 + 4E^2}. \quad (6.23)$$

In the limit  $\max(\omega, T) \ll E_{Th}$  the result  $L_{\text{eff}} = L_x$  is uncovered, whereas in the opposite case,  $\max(\omega, T) \gg E_{Th}$ , one gets

$$L_{\text{eff}}(\omega, T) = \frac{\sqrt{D}}{2\omega} \int_0^{\infty} \left[ \tanh \frac{E_+}{2T} - \tanh \frac{E_-}{2T} \right] \frac{dE}{\sqrt{E}} = \begin{cases} \sqrt{2D/|\omega|}, & \text{if } T \ll \omega; \\ 0.95\sqrt{D/2T}, & \text{if } \omega \ll T. \end{cases} \quad (6.24)$$

The number 0.95 in the above equation is the approximate value for  $(1 - 2^{-3/2})\pi^{-1/2}\zeta(3/2)$ .

Representation (6.21) can also be used in order to derive an expression for the *nonlinear* Andreev current  $I_A(V)$ . We present this derivation in Appendix B, the result is that dc subgap current  $I_A(V) = VG_A(2eV)$ , where the function  $G_A(\omega)$  is defined in Eq. (6.22). This relation between frequency-dependent linear response and static nonlinear response is due to the fact that static voltage  $V$  applied to the superconductor leads to the oscillation of its order parameter  $|\Delta|e^{i\theta}$  with the frequency  $2eV$ .

To summarize this subsection, we emphasize that within the usual semiclassical approximation the term describing Andreev conductance is seen as a result of Gaussian integration over noninteracting Cooperon modes in the N metal. Below we will take into account Cooperon nonlinearities which are due to interactions in the Cooper and direct Coulomb channels. Basically, there are two different kinds of the interaction effects: the first one is due to the presence of the electron-electron interaction in the Cooper channel (which is itself renormalized in a way dependent on the degree of disorder, as shown by Finkelstein), whereas the second one is of the same nature as the zero-bias anomaly in the usual tunneling conductance.<sup>7</sup> These two effects are intrinsically different, as the first one is determined by diffusion modes with low frequencies  $\omega \ll Dq^2$  and does not depend on the long-range tail of the Coulomb potential, whereas the

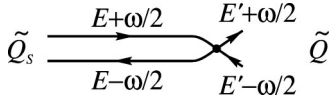


FIG. 5. Correction to the interface transparency  $\gamma$  due to interaction in the Cooper channel.

second one comes from relatively high-frequency fluctuations with  $\omega \gg Dq^2$  and *does depend* on the actual behavior of the Coulomb potential at large space scales (and thus it depends on the sample geometry as well). Below we start from the study of the first (“Finkelstein’s”) effect and later on will include the effect of the zero-bias anomaly. The latter effects may indeed be strongly suppressed if a good conductor is placed near a dirty metal film, so that the Coulomb interaction in the film becomes screened on a relatively short distance [cf. Eq. (4.24) and Refs. 7,41].

## 2. The effect of interactions in the Cooper channel

In this subsection we will show that the Cooper-channel interaction in the normal metal leads to a logarithmic (or weak power-law) dependence of the Andreev conductance on frequency. To take it into account, one has to sum all the inclusions of the vertex  $S_\lambda$  into the Cooperon propagator. Such a summation can be effectively expressed as a renormalization of the tunneling action (6.1) due to the presence of the Cooper repulsion  $\lambda$ . As a result, the tunneling conductance  $\gamma$  gets renormalized down to  $\gamma(\zeta)$  which should then be substituted into Eq. (6.22) for  $G_A$ .

Logarithmic corrections to the boundary term (6.1) originate from pairing with the Cooper nonlinearity vertex  $S_\lambda$  given by Eq. (4.16):

$$\Delta S_\gamma = i \langle S_\gamma S_\lambda \rangle. \quad (6.25)$$

The diagram is shown in Fig. 5, and contains one fast Cooperon mode connecting  $S_\gamma$  and  $S_\lambda$ :

$$\begin{aligned} \langle S_\gamma S_\lambda \rangle &= -\frac{i\pi^3 \nu \gamma \lambda}{16} \langle \text{Tr}_\Gamma \tilde{Q}_S \sigma_z \tau_z \mathcal{W}' \\ &\quad \times \text{Tr}\{\sigma_x, \tilde{Q}\} u \sigma_z [\tau_z, \mathcal{W}'] u \rangle, \end{aligned} \quad (6.26)$$

where  $Q_S = u Q_S u$ , according to the general rule (4.4).

Consider first the spacial structure of Eq. (6.26). The first trace corresponding to  $S_\gamma$  is taken along the N-S interface  $\Gamma$ , whereas the second one corresponding to  $S_\lambda$  is taken in the bulk of the N region at a distance not larger than  $\sqrt{D/\Omega_*}$  from the interface. That is, from the point of view of slow variables, both  $S_\gamma$  and  $S_\lambda$  are taken at the same point on the boundary  $\Gamma$ . In virtue of the coordinate integration in  $S_\lambda$ , the correlator  $\langle \mathcal{W}' \mathcal{W}' \rangle$  must be taken at zero momentum, so that the energy  $E$  becomes the only fast variable involved. Furthermore, due to the presence of the commutator of  $\mathcal{W}'$  with  $\tau_z$  under the second trace in Eq. (6.26), only Cooperon modes give a nonzero contribution [cf. Eq. (4.10)]. Then, from the first trace one concludes that the whole average (6.26) does not vanish only in the subgap region,  $\Omega < \Delta$ . In other words, this reflects the trivial fact that the metallic tunneling conductance is not renormalized by interaction with Cooperons. In the deep subgap limit, the superconduc-

tive  $Q_S$  is given by Eq. (6.5), it is independent of the “center-of-mass” energy  $E$  and depends on  $\omega$  through the rotating phase  $\theta(t)$ .

Employing the Cooperon-related part of the contraction rule (4.14), one obtains

$$\begin{aligned} \langle S_\gamma S_\lambda \rangle &= -\frac{\pi^2 \gamma \lambda}{32} \int_\Gamma d\mathbf{r} \int \frac{d\omega dE dE'}{(2\pi)^3} \frac{1}{E} \\ &\quad \times \text{tr}(AB + A\sigma_z B\sigma_z - A\tau_z B\tau_z - A\sigma_z \tau_z B\sigma_z \tau_z) \sigma_z, \end{aligned} \quad (6.27)$$

where  $\Omega_* < E < \Omega$ , whereas  $\omega, E' < \Omega_*$ , and the matrices

$$A = u_E Q_S(\omega) u_E \sigma_z \tau_z,$$

$$B = u_E \{\sigma_x, Q_{E' - \omega/2, E' + \omega/2}\} u_E \sigma_z \tau_z$$

are taken at the same point at the interface (to logarithmic accuracy, all fast energies are equal to  $E$ ).

Using the property  $\{Q_S, \tau_z\} = 0$  and the relation  $\Lambda_0 = u \sigma_z u$ , we transform Eq. (6.27) as

$$\langle S_\gamma S_\lambda \rangle = \frac{\pi^2 \gamma \lambda}{16} \int_\Gamma d\mathbf{r} \int \frac{d\omega dE dE'}{(2\pi)^3} \frac{1}{E} \text{tr}\{Q_S, \{\sigma_x, Q\}\} \Lambda_0(E). \quad (6.28)$$

The matrix  $\Lambda_0(E)$  in Eq. (6.28) can be replaced by  $2\sigma_+ \tanh(E/2T)$ , since contributions of its other components vanish by parity after integration over  $E$ . Employing the relations  $Q_S \propto 1_K$  and  $\{\sigma_+, \sigma_x\} = 1$ , one arrives at

$$\langle S_\gamma S_\lambda \rangle = \frac{\pi \gamma \lambda}{4} \ln \frac{\Omega}{\Omega_*} \text{Tr}_\Gamma Q_S Q. \quad (6.29)$$

This term effectively renormalizes the interface transparency; substituting Eq. (6.29) into Eq. (6.25), we obtain the RG equation for  $\gamma$ :

$$\frac{\partial \gamma}{\partial \zeta} = -\gamma \lambda. \quad (6.30)$$

A few comments are in order concerning this equation. First of all, we remind the reader that it is valid in the subgap region  $\zeta > \zeta_\Delta$ , where  $\zeta_\Delta = \ln 1/\Delta \tau$ ; for higher energies  $\gamma$  remains unaffected. In the derivation we assumed that both  $\omega$  and  $T$  are small compared to the running cutoff  $\Omega$ , otherwise the RG should stop at the scale  $\max(\omega, T)$ .

Equation (6.30) contains the running coupling constant  $\lambda(\zeta)$  which is determined by Eq. (5.29) with  $\lambda_0 < 0$  replaced by  $\lambda_n > -\lambda_g$ , a Cooper-channel repulsion in the N metal, defined at the energy scale  $\Delta$ . Looking at Eq. (6.30) one can think that it is valid for any geometry of the system since it arises as a result of integration over energy. This is, however, not the case and the RG exists in 2D only. The point is that the coupling constant  $\lambda(\zeta)$  logarithmically depends on the scale in 2D case only (cf. Sec. V C). Recently a method to treat  $T_c$  suppression for dirty superconducting (SC) films in the 1D–2D crossover region was developed,<sup>43</sup> which does not employ the existence of logarithmic RG equations.

Substituting  $\lambda(\zeta)$  from Eq. (5.27), we write down the solution of Eq. (6.30) with the boundary condition  $\gamma(\zeta_\Delta) = \gamma_0$ :



$$\gamma(\zeta) = \frac{\gamma_0}{\left(1 + \frac{\lambda_n}{\lambda_g} \tanh \lambda_g(\zeta - \zeta_\Delta)\right) \cosh \lambda_g(\zeta - \zeta_\Delta)}. \quad (6.31)$$

Now this equation can be used to find the Andreev conductance modified by the interaction in the Cooper channel. In this regard we note that all integrals in Eq. (6.17) are not logarithmic. Consequently, with logarithmic accuracy, it is sufficient to use the semiclassical expression for  $G_A$ , but with the renormalized barrier transparency:

$$G_A = \frac{\gamma^2 \left( \ln \frac{1}{\Omega \tau} \right)}{\gamma_0^2} G_A^{(0)}, \quad (6.32)$$

where the semiclassical value of  $G_A^{(0)}$  is given by Eq. (6.22) and the low-energy RG cutoff  $\Omega = \max(D/L_x^2, \omega, eV, T, \tau_\phi^{-1})$ , with  $\tau_\phi$  being the electron decoherence time in the N metal (all these quantities are assumed to be much below  $\Delta$ ). Substituting  $\gamma(\zeta)$  from Eq. (6.31) one obtains

$$G_A = \frac{G_T^2 L_{\text{eff}}(L_x, \omega, T)}{g L_y} \times \frac{4(\Omega/\Delta)^{2\lambda_g}}{[(1 + \lambda_n/\lambda_g) + (1 - \lambda_n/\lambda_g)(\Omega/\Delta)^{2\lambda_g}]^2}, \quad (6.33)$$

where  $\lambda_g$  is defined in Eq. (5.28). This equation is valid provided that the problem is effectively two-dimensional, cf. discussion after Eq. (6.30). This condition is satisfied as long as  $L_\Omega \leq L_y$ , where  $L_\Omega = \sqrt{D/\Omega}$ . According to Eq. (6.33), in the low- $\Omega$  limit the Andreev conductance acquires an anomalous power-law suppression with an exponent  $2\lambda_g$ . The total power-law exponent, describing the growth of  $G_A(\Omega)$  with the  $\Omega$  decrease, is equal to

$$x_A = \frac{1}{2} - 2\lambda_g = \frac{1}{2} \left( 1 - \sqrt{\frac{4e^2}{\pi^2 \hbar} R_\square} \right). \quad (6.34)$$

#### D. Small SC island in contact with 2D film

Here we consider another example of an N-S contact, namely, a contact of a small superconductive island of size  $d$  with a thin dirty normal film of linear size  $L \gg d$ , shown in Fig. 6. The edge of the film is connected to a normal reservoir. The semiclassical result for the Andreev conductance in this geometry can be inferred from the general relation  $\sigma_A^{\text{cl}} = \sigma_T^2 R_D$ , where the metal resistance  $R_D$  is now a ‘‘spreading’’ resistance which grows logarithmically with the film size:  $R_D = (1/2 \pi \sigma) \ln(L/d)$ . If temperature or frequency is larger than the Thouless energy scale  $D/L^2$ , the shortest of the effective lengths  $L_{T,\omega} = \sqrt{D/(T,\omega)}$  should be used instead of  $L$ . Logarithmic growth of the Andreev conductance with the space scale implies that  $G_A$  itself becomes a running coupling constant subject to the RG procedure. Below we will derive and solve the RG equation for the Andreev conductance in the presence of the Cooper interaction renormalized by the Coulomb repulsion (still we will not touch here

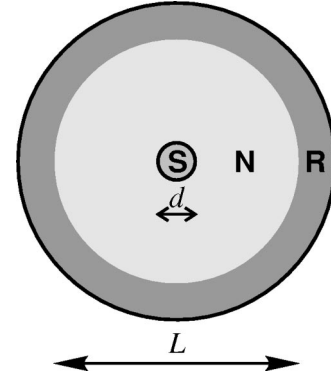


FIG. 6. Small superconductive island (S) of size  $d$  connected to a reservoir (R) through a dirty normal film (N) of size  $L \gg d$ .

the effect of the tunneling DOS suppression; it will be considered below, in the next subsection).

Similarly to Eq. (6.13), the correction to the Andreev term (6.4) in the action comes from the averaging of two boundary vertices  $S_\gamma$ :

$$\Delta S_A = \frac{i}{2} \langle S_\gamma^2 \rangle. \quad (6.35)$$

At scales larger than its size  $d$  the SC island can be considered as a point object. Then the boundary term (6.1) can be written in a local form:  $S_\gamma = -(i\pi/4) G_T \text{Tr} Q_S Q(0)$ , with  $G_T = \mathcal{A} \gamma$ , where  $\mathcal{A}$  is the area of the contact between the island and the film. The relevant pairing in Eq. (6.35),

$$\langle S_\gamma^2 \rangle = - \frac{\pi^2 G_T^2}{16} \langle [\text{Tr} \tilde{U} Q_S \tilde{U}^{-1} \sigma_z \tau_z \mathcal{W}']^2 \rangle, \quad (6.36)$$

is shown diagrammatically in Fig. 7. Here all energies are coupled to the slow matrices  $\tilde{U}$ , and the internal Cooperon momentum is the only fast variable. Integrating out fast modes with the help of the contraction rule (4.15), one obtains

$$\langle S_\gamma^2 \rangle = \frac{\pi G_T^2}{16\nu} \int \frac{d\epsilon_1 d\epsilon_2 d\mathbf{q}}{(2\pi)^4} \frac{1}{Dq^2} \text{Tr}(AB - A \sigma_z \tau_z B \sigma_z \tau_z), \quad (6.37)$$

where (all fields are taken at the point  $\mathbf{r}=0$ )

$$A = (\tilde{U} Q_S \tilde{U}^{-1})_{\epsilon_1, \epsilon_2} \sigma_z \tau_z,$$

$$B = (\tilde{U} Q_S \tilde{U}^{-1})_{\epsilon_2, \epsilon_1} \sigma_z \tau_z.$$

Only the first term under the trace in Eq. (6.37) is important. Evaluating the logarithmic integral over  $\mathbf{q}$ , one finds

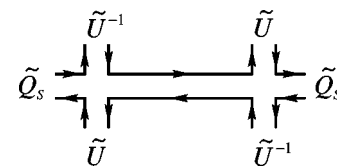


FIG. 7. Correction to the Andreev term  $S_A$ .

$$\langle S_\gamma^2 \rangle = \frac{G_T^2}{8g} \ln \frac{\Omega}{\Omega_*} \text{Tr}(Q_S Q)^2. \quad (6.38)$$

Equation (6.38) is valid at space scales  $L$  larger than the size of the island  $d$ , i.e., for energies  $\Omega \ll \omega_d = D/d^2$ . It shows that the RG procedure generates the term in the action of the form

$$S_A = \frac{i\pi}{16} G_A \text{Tr}(Q_S Q)^2, \quad (6.39)$$

which reduces to Eq. (6.4) when all fast modes of  $Q$  are integrated out and  $Q$  is replaced by  $\Lambda$ . The running constant  $G_A$  obeys the following RG equation:

$$\frac{\partial G_A}{\partial \zeta} = \frac{G_T^2}{4\pi g}, \quad (6.40)$$

valid for  $\zeta > \zeta_d = \ln 1/\omega_d \tau$ .

In the semiclassical limit,  $\gamma \equiv G_T/\mathcal{A}$  is constant and integrating Eq. (6.40) over  $\zeta = 2 \ln(L/l)$  one reveals the above-mentioned result,  $G_A^{\text{cl}} = (G_T^2/2\pi g) \ln(L/d)$ . Beyond semiclassicals, Eq. (6.40) becomes nontrivial since fluctuations which renormalize the tunneling conductance  $\gamma(\zeta)$  must be taken into account. Below we will assume that  $\omega_d \sim \Delta$  and thereby neglect small corrections of the order of  $\lambda_g |\zeta_\Delta - \zeta_d|$ . Then, substituting  $\gamma(\zeta)$  from Eq. (6.31) and integrating Eq. (6.40) over  $\zeta$  between  $\zeta_d$  and  $\ln(1/\Omega \tau)$ , we obtain the result for  $G_A(\Omega)$ :

$$G_A(\Omega) = \frac{G_T^2}{4\pi g} \frac{1 - (\Omega/\Delta)^{2\lambda_g}}{(\lambda_g + \lambda_n) + (\lambda_g - \lambda_n)(\Omega/\Delta)^{2\lambda_g}}. \quad (6.41)$$

A general feature of expression (6.41) is that  $G_A(\Omega)$  approaches a constant value,  $G_T^2/4\pi g(\lambda_n + \lambda_g)$ , in the low- $\Omega$  limit. The semiclassical expression<sup>42</sup> for the Andreev conductance,  $G_A^{\text{cl}}(\Omega) = (G_T^2/4\pi g) \ln(\omega_d/\Omega)$ , predicts growth of  $G_A(\Omega)$  with the increase of the relevant space scale  $L_\Omega = \sqrt{D/\Omega}$ , due to the growth of the region where the electron and Andreev-reflected hole interfere constructively. This expression for  $G_A^{\text{cl}}(\Omega)$  follows from Eq. (6.41) in the limit  $\lambda_n \rightarrow 0$  and  $\ln(\Delta/\Omega) \ll \sqrt{g}$ . Our result (6.41) demonstrates that (in the present geometry when the current flow is effectively two-dimensional) the Cooper-channel repulsion acts as if it imposes an upper limit for the time duration of such a constructive interference. The same qualitative behavior would be seen in the absence of Finkelstein's corrections as well: taking first the limit  $g \gg \ln^2(\Delta/\Omega)$  in Eq. (6.41), one would find  $G_A \approx G_T^2/4\pi g \lambda_n$  as  $\Omega \rightarrow 0$  (cf. Ref. 44 where a similar expression was used for  $G_A$ ; note, however, that a numerical mistake was made in Ref. 44, which lead to the overestimation of  $G_A$  by a factor of 2). Below we will see that the effect of the Coulomb-blockade suppression of tunneling is even more drastic, as it leads to the decrease of the Andreev conductance at  $\Omega \rightarrow 0$ .

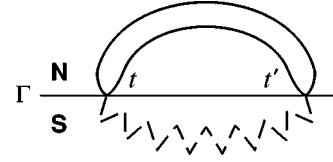


FIG. 8. An example of factorization of the ZBA-type fluctuations for the Andreev conductance. The dashed zigzag line denotes the correlator  $\langle\langle e^{iK(t)\tau_z} \rangle\rangle_i \langle\langle e^{iK(t')\tau_z} \rangle\rangle_j$ .

## E. Effect of the zero-bias anomaly

### 1. General treatment

In the preceding consideration of the Andreev conductivity we neglected the effect of high-frequency ( $Dq^2 \ll \omega$ ) Coulomb fluctuations. They are responsible for the zero-bias anomaly (ZBA) in the usual tunneling conductance (calculated originally by Altshuler and Aronov<sup>7</sup>), which, in 2D, leads to a suppression of tunneling by the relative order of  $g^{-1} \ln^2(1/\Omega \tau)$ . Later it was argued by Finkelstein<sup>12</sup> and shown (within semiclassical approach) by Levitov and Shytov<sup>41</sup> that the first correction must be exponentiated to get a result valid in the low-frequency limit as well. Recently, Kamenev and Andreev<sup>22</sup> rederived this result microscopically using the Keldysh  $\sigma$ -model approach and separating the Coulomb phase  $\vec{K}$  according to Eq. (3.12).

After the gauge transformation (3.12), the boundary tunneling term (6.1) acquires the form

$$S_\gamma = -\frac{i\pi}{4} \gamma \text{Tr}_\Gamma e^{i\vec{K}_{12}\tau_z Q^{(1)}} e^{-i\vec{K}_{12}\tau_z Q^{(2)}}, \quad (6.42)$$

where  $\vec{K}_{12} = \vec{K}^{(1)} - \vec{K}^{(2)}$  is the Coulomb phase difference across the interface. In our study of the Andreev conductance, we will assume that the superconductor is connected to a low-impedance external environment which keeps fixed its average electric potential, and neglect the Coulomb phase  $\vec{K}_S$  in the superconductor for the reasons explained in the last paragraph of Sec. IV C. Hence, the ZBA is determined by the fluctuations of the normal-metal phase  $\vec{K}$ .

On singling out the Coulomb phase by the transformation (3.12), the calculation of the ZBA becomes very simple:<sup>22</sup> one has just to average the phase factors  $e^{\pm i\vec{K}}$  in the boundary term (6.42) over the Gaussian fluctuations of  $\vec{K}$  with the correlator (4.30). Indeed,  $\langle K_i K_j \rangle$ , being integrated over frequency and momentum, yields the aforementioned  $g^{-1} \ln^2(1/\Omega \tau)$ . On the other hand, since the bulk action (4.5) depends only on derivatives of  $\vec{K}$ , any  $\langle K_i K_j \rangle$  pairing in the bulk contains an additional power of frequency or momentum, its contribution is less singular than  $\ln^2(1/\Omega \tau)$  in the limit  $\Omega \rightarrow 0$ , and therefore can be neglected.

From this general observation it follows that in calculating the Andreev conductance the effect of the interaction in the Cooper channel is factorized from the ZBA effect. In other words, the full Cooperon which determines the subgap conductance is a product (in the time domain) of the Cooperon without the ZBA (which was studied in Secs. VI C and VI D) and the ZBA factor  $Z^2(t)$  [cf. Eq. (7.15) below]. Such a factorization is shown diagrammatically in Fig. 8. Our previous analysis implied  $Z(t) = 1$ , i.e., weak ZBA at the scale

$\Omega \sim \tau^{-1} e^{2\pi\sqrt{g}}$  relevant for the saturation of the RG flow for the Cooper repulsion constant  $\lambda$ , cf. Eq. (5.29). We will use this factorization property explicitly in the study of the Josephson proximity coupling in Sec. VII. In the calculation of the Andreev conductivity we will rather follow another approach: the effect of the ZBA will be taken into account within the RG scheme together with the Cooper channel renormalization described above.

## 2. Rectangular N-S contact

High-frequency ZBA-type fluctuations of the Coulomb phase  $\vec{K}$  in Eq. (6.42) lead to an additional suppression of the effective transparency  $\gamma$  of the interface. This effect is different in the metallic ( $\Delta < \Omega < \tau^{-1}$ ) and superconducting ( $\Omega < \Delta$ ) regions. In the metallic case, the ZBA is governed by a single-electron tunneling, which is the only source of the transparency suppression, whereas in the superconducting case, the reduction of  $\gamma$  becomes more pronounced due to a coherent tunneling of Cooper pairs, carrying the double charge.

The renormalized value of  $\gamma$  can be extracted from the boundary action (6.42) after averaging over fast fluctuations of the Coulomb phase  $\vec{K}'$  in the metal:

$$\langle S_\gamma \rangle = -\frac{i\pi}{4} \gamma \text{Tr}_\Gamma \langle e^{-iK'_1(t_1)\tau_z} \tilde{Q}_S(t_1, t_2) e^{iK'_1(t_2)\tau_z} \tilde{Q}(t_2, t_1) \rangle, \quad (6.43)$$

where we neglected the Coulomb phase  $\vec{K}_S$  in the superconductor as discussed above and used the fact that only the correlator of classical components of the phase  $\vec{K}$  may be retained with logarithmic accuracy.<sup>22</sup> On eliminating fast degrees of freedom, the term  $S_\gamma$  will reproduce itself with the modified value of the transparency  $\gamma$  (cf. Ref. 40). Further transformation of Eq. (6.43) depends on the structure of the matrix  $Q_S$  in the Nambu and time spaces and on the geometry of the interface.

In the metallic energy range,  $\Delta < \Omega$ , the off-diagonal in the Nambu space components of the superconductive  $Q_S$  can be neglected [cf. Eq. (3.10)], so that it commutes with the phase factor  $e^{iK_1(t)\tau_z}$ . Then the perturbative correction to  $\gamma$  can be deduced from Eq. (6.43):

$$\Delta\gamma = \gamma [e^{-1/2\langle [K'_1(t_1) - K'_1(t_2)]^2 \rangle} - 1]. \quad (6.44)$$

The fast phases  $K'_1$  are correlated only at small times  $\leq \Omega_*^{-1}$ , while the time difference in Eq. (6.44) is large enough,  $|t_1 - t_2| \geq \Omega_*$ , due to an implicit  $\epsilon$  dependence of  $Q_S$  at the metallic region. Therefore, the cross average in Eq. (6.44) can be neglected, and expanding the exponent one arrives at

$$\Delta\gamma = -\gamma \langle [K'_1(0)]^2 \rangle. \quad (6.45)$$

The value of the correlator  $\langle [K'_1(0)]^2 \rangle$  is not universal and depends on the setup considered. For the tunneling into the whole 2D plane (that corresponds to the geometry of a small SC island shown in Fig. 6), one can employ Eq. (4.30):

$$\begin{aligned} & \langle [K'_1(0)]^2 \rangle_{\text{plane}} \\ &= \int_{\Omega_*}^{\Omega} \frac{d\omega}{\pi} \int \frac{d\mathbf{q}}{(2\pi)^2} \\ & \times \text{Im} \frac{V_0(q)}{(Dq^2 - i\omega)(2\nu V_0(q)Dq^2 + Dq^2 - i\omega)}. \end{aligned} \quad (6.46)$$

The leading double-log contribution to this expression comes from the region  $Dq^2 \ll \omega \ll \nu V_0(q)Dq^2$  where the integrand is given by  $1/(g\omega q^2)$ . Note that in this limit of strong Coulomb interaction, a specific form of the potential  $V_0(q)$  drops from the integrand and enters the result only through the cutoff of logarithmic  $q$  integration. For the case of the bare 2D Coulomb potential (4.23), one obtains

$$\begin{aligned} \langle [K'_1(0)]^2 \rangle_{\text{plane}} &= \frac{1}{2\pi^2 g} \int_{\Omega_*}^{\Omega} \frac{d\omega}{\omega} \int_{\omega/2\pi\sigma}^{\sqrt{\omega/D}} \frac{dq}{q} \\ &= \frac{1}{8\pi^2 g} \left( \ln^2 \frac{\omega_0}{\Omega_*} - \ln^2 \frac{\omega_0}{\Omega} \right), \end{aligned} \quad (6.47)$$

where  $\omega_0 = (2\pi\sigma)^2/D$ .

In the rectangular geometry of Fig. 4, a charge tunnels into the edge of the half-plane. A similar problem for the half-space was considered in Ref. 45. Though a complete treatment of such problems is involved, a double-log asymptotics can be easily derived. To find it, one should use the fact that in the relevant region,  $Dq^2 \ll \omega \ll 2\pi\sigma q$ , the screened Coulomb interaction is determined solely by the inverse density-density correlator<sup>45</sup> whereas the bare Coulomb potential drops from equations as we have seen above. The density-density correlator can be obtained with the help of the eigenfunctions of the diffusion equation with the proper boundary conditions. For the rectangular geometry such eigenfunctions are given by Eq. (6.16). Then the value of  $\langle [K'_1(0)]^2 \rangle$  can be obtained analogously to Eq. (6.47); it depends on the distance  $r$  from the edge of the half-plane and is given by

$$\begin{aligned} \langle [K'_1(0)]^2 \rangle_{\text{half-plane}} &= \frac{1}{\pi^2 g} \int_{\Omega_*}^{\Omega} \frac{d\omega}{\omega} \\ & \times \int_{\omega/2\pi\sigma}^{\sqrt{\omega/D}} \frac{d\mathbf{q}}{2\pi q^2} \cos^2(q_x r), \end{aligned} \quad (6.48)$$

where the infinite half-plane ( $L_x, L_y \rightarrow \infty$ ) is implied. For large  $r$ , the cosine squared in Eq. (6.48) can be substituted by its average value, 1/2, that leads to the infinite-plane result (6.47). For the tunneling directly into the edge ( $d=r$ ) the correlator (6.48) appears to be two times larger.

Hence the RG equation for the transparency  $\gamma$  of a flat N-S boundary in the metallic energy range ( $\zeta < \zeta_\Delta$ ) reads

$$\frac{\partial\gamma}{\partial\zeta} = -\frac{\zeta - \zeta_0}{2\pi^2 g} \gamma, \quad (6.49)$$

where  $\zeta_0$  is defined at the scale  $\omega_0$ ,  $\zeta_0 = \ln 1/\omega_0\tau$ . Being integrated over  $\zeta$  from 0 to  $\ln(1/\Omega\tau)$ , it yields exponential

reduction of the tunneling conductance (which is two times stronger than the result<sup>12,41,22</sup> for the infinite plane):

$$\gamma(\zeta) = \gamma_0 Z_M(\zeta), \quad (6.50)$$

where the factor  $Z_M(\zeta)$  is given by

$$\begin{aligned} Z_M(\zeta) &= \exp\left(-\frac{(\zeta - \zeta_0)^2 - \zeta_0^2}{4\pi^2 g}\right) \\ &= \exp\left(-\frac{1}{4\pi^2 g} \ln \frac{1}{\Omega \tau} \cdot \ln \frac{\omega_0^2 \tau}{\Omega}\right). \end{aligned} \quad (6.51)$$

When the RG cutoff  $\Omega$  becomes smaller than  $\Delta$ , the situation changes. First of all,  $Q_S$  acquires an  $\epsilon$ -independent form (6.5), i.e., it becomes a  $\delta$  function in the time domain,  $t_1 = t_2$ . It also anticommutes with  $e^{iK_1(t)\tau_z}$  due to its structure in the Nambu space. In this limit, the Coulomb phase  $\vec{K}$  can be considered as an additive correction to the SC phase  $\vec{\theta}$  on the island, with the total effective phase  $\vec{\theta}_{\text{eff}} = \vec{\theta} - 2\vec{K}$ . Then the effect of the ZBA in the subgap limit can be attributed to the high-frequency fluctuations of  $\vec{\theta}_{\text{eff}}(t)$ , destroying the Cooper pair coherence at large time scales. Repeating the steps that lead to Eqs. (6.44) and (6.45), one has instead

$$\Delta \gamma = \gamma [e^{-1/2(2K_1'(t))^2} - 1] = -2\gamma [K_1'(0)]^2. \quad (6.52)$$

The effect of the ZBA anomaly should be taken into account together with the renormalization of the transparency due to interactions in the Cooper channel, cf. Eq. (6.30). Thus, we get the following RG equation for  $\gamma$  valid in the subgap limit  $\zeta > \zeta_\Delta$ :

$$\frac{\partial \gamma}{\partial \zeta} = -\lambda \gamma - \frac{\zeta - \zeta_0}{\pi^2 g} \gamma. \quad (6.53)$$

Solving this equation for  $\zeta > \zeta_\Delta$ , we obtain a modification of Eq. (6.31):

$$\gamma(\zeta) = \frac{\gamma_\Delta Z_S(\zeta)}{\left(1 + \frac{\lambda_n}{\lambda_g} \tanh \lambda_g (\zeta - \zeta_\Delta)\right) \cosh \lambda_g (\zeta - \zeta_\Delta)}, \quad (6.54)$$

where  $\gamma_\Delta \equiv \gamma(\zeta_\Delta) = \gamma_0 Z_M(\zeta_\Delta)$  is the transparency renormalized by the ZBA in the metallic energy region  $\Delta < \Omega < \tau^{-1}$ , and the multiplicative factor  $Z_S(\zeta)$  accounts for the subgap ZBA effect:

$$\begin{aligned} Z_S(\zeta) &= \exp\left(-\frac{(\zeta - \zeta_0)^2 - (\zeta_\Delta - \zeta_0)^2}{2\pi^2 g}\right) \\ &= \exp\left(-\frac{1}{2\pi^2 g} \ln \frac{\Delta}{\Omega} \cdot \ln \frac{\omega_0^2}{\Delta \Omega}\right). \end{aligned} \quad (6.55)$$

So far we considered an infinite half-plane with  $L_x, L_y \rightarrow \infty$ . In analogy with the treatment in Sec. VI C 2, it might be tempting to substitute the renormalized transparency  $\gamma(\ln(1/\Omega\tau))$  given by Eq. (6.54) into Eq. (6.22) to get the Andreev conductance  $G_A(\Omega)$  in the rectangular geometry.

This, however, would be wrong. The point is that the expression (6.48) for  $\langle [K'(0)]^2 \rangle$  contains two coupled integrations over  $\omega$  and  $q$ , with *two* frequency-dependent length scales,  $\sqrt{D/\omega}$  and  $2\pi\sigma/\omega$ . The former is the usual diffusive length that shows how far a charge spreads during time  $\omega^{-1}$ . The latter is associated with the electric field propagation in a conducting medium. One logarithm in the double-log ZBA expression comes from the spacial region  $\sqrt{D/\omega} < R < 2\pi\sigma/\omega$ . However, for small enough  $\omega$ , the length  $2\pi\sigma/\omega$  becomes larger than the system size  $L_x$  (it is assumed that  $L_x \leq L_y$ ). This finite-size effect can be accounted for in Eq. (6.48) by substituting  $\max(\omega/2\pi\sigma, 1/L_x)$  as a lower limit of  $q$  integration. Physically this procedure means that we neglect external impedance of the circuit connected to our dirty film in comparison with the effective ‘‘spreading resistance’’ of the film [given by  $\pi r(\zeta)$ , where function  $r(\zeta)$  is determined below in Eq. (6.58)]. It is then straightforward to generalize the RG equations (6.49) and (6.53) for finite systems. In the metallic energy region the transparency obeys

$$\frac{\partial \gamma}{\partial \zeta} = -r(\zeta) \gamma, \quad (6.56)$$

while in the subgap limit one has

$$\frac{\partial \gamma}{\partial \zeta} = -\lambda \gamma - 2r(\zeta) \gamma. \quad (6.57)$$

The function  $r(\zeta)$  is defined as

$$r(\zeta) = \frac{1}{2\pi^2 g} \times \begin{cases} \zeta - \zeta_0, & \text{if } \Omega > \omega_\sigma; \\ \zeta_{Th} - \zeta, & \text{if } \omega_\sigma > \Omega > E_{Th}; \\ 0, & \text{if } E_{Th} > \Omega, \end{cases} \quad (6.58)$$

where  $\omega_\sigma = 2\pi\sigma/L_x$  and  $\zeta_{Th} = \ln 1/E_{Th}\tau = 2 \ln(L_x/l)$ .

The solutions of Eqs. (6.56), (6.57) can also be represented in the form (6.50), (6.54) with the modified functions  $Z_M(\zeta)$  and  $Z_S(\zeta)$ . We will not present here the complete list of formulas for arbitrary values of  $\omega_\sigma/\Delta$  and  $\omega_\sigma\tau$  but will focus instead on  $Z_S(\zeta)$  in the experimentally relevant case. If  $L_x$  is measured in  $\mu\text{m}$  then the energy  $\omega_\sigma$  (in Kelvins) is given by  $100g/L_x$ , which appears to be greater than  $\Delta$  for a reasonable experimental setup. Hence, in studying the subgap frequency region, the lower  $q$  cutoff is given by the inverse system size,  $L^{-1}$ , and any information about the Coulomb potential  $V_0(q)$  drops from the resulting expression. Solving then Eq. (6.57) one obtains

$$\begin{aligned} Z_S(\zeta) &= \exp\left(-\frac{(\zeta - \zeta_\Delta)[\zeta_{Th} - (\zeta + \zeta_\Delta)/2]}{2\pi^2 g}\right) \\ &= \exp\left(-\frac{1}{2\pi^2 g} \ln \frac{\Delta}{\Omega} \cdot \ln \frac{\Delta \Omega}{E_{Th}^2}\right). \end{aligned} \quad (6.59)$$

Note that  $\Omega = \max(E_{Th}, \omega, eV, T, \tau_\phi^{-1})$  and cannot be smaller than  $E_{Th}$ .

Using Eq. (6.59) and substituting  $\gamma(\zeta)$  from Eq. (6.54) into Eq. (6.22), we obtain the resulting expression:

$$G_A = \frac{G_{T\Delta}^2 L_{\text{eff}}(L_x, \omega, T)}{g L_y} \times \frac{4(\Omega/\Delta)^{2\lambda_g}}{[(1 + \lambda_n/\lambda_g) + (1 - \lambda_n/\lambda_g)(\Omega/\Delta)^{2\lambda_g}]^2} \times \exp\left(-\frac{1}{\pi^2 g} \ln \frac{\Delta}{\Omega} \cdot \ln \frac{\Delta \Omega}{E_{Th}^2}\right), \quad (6.60)$$

where  $G_{T\Delta} \equiv G_{TZ_M}(\zeta_\Delta)$  stands for the single-particle tunneling conductance at the scale  $\Delta$  renormalized by the normal-metal ZBA. Equation (6.60) is one of the main results of this paper, it shows that the original growth of  $G_A(\Omega) \propto \Omega^{-x_A}$  with the  $\Omega$  decrease [the exponent  $x_A$  is defined in Eq. (6.34)] stops at  $\ln(\Delta/\Omega) \sim \sqrt{g}$ , and at lower  $\Omega$  the Andreev conductance decreases due to the zero-bias anomaly. In the intermediate frequency range  $\ln(\Delta/\Omega) \sim \sqrt{g}$  both Finkelstein's effect and the ZBA effect are of the same importance, whereas in the infrared limit the ZBA effect is the most important one. The influence of the last effect upon the Andreev conductance was predicted by Huck, Hekking, and Kramer<sup>28</sup> on phenomenological grounds. They considered a N-S junction coupled to the model dissipative environment characterized by some impedance  $\mathcal{Z}(\omega)$ . We provide here a microscopic calculation of this effective impedance:  $\mathcal{Z}(\omega) = \pi r(\ln(1/\Omega\tau))$ , with the function  $r(\zeta)$  defined in Eqs. (6.58) and (6.64) below.

Until now we considered the case of unscreened bare Coulomb potential  $V_0(q) = 2\pi e^2/q$ . In the presence of an additional (clean) metal gate, the Coulomb potential in the dirty metal layer changes according to Eq. (4.24). As a result,  $V_0^{\text{scr}}(0) = 4\pi e^2 b$ , which modifies the law of propagation of the electric field, which now becomes diffusive with the effective diffusion coefficient  $D_* = 8\pi v e^2 b \cdot D$ . The ratio  $D_*/D = 2\kappa_2 b \gg 1$ , where  $\kappa_2 = 4\pi v e^2$  is the inverse 2D screening radius. In a finite-size system, Eq. (6.58) should be modified as

$$r^{\text{scr}}(\zeta) = \frac{1}{2\pi^2 g} \times \begin{cases} \ln \frac{D_*}{D}, & \text{if } \frac{D}{D_*} \Omega > E_{Th}; \\ \zeta_{Th} - \zeta, & \text{if } \Omega > E_{Th} > \frac{D}{D_*} \Omega; \\ 0, & \text{if } E_{Th} > \Omega. \end{cases} \quad (6.61)$$

In an effectively infinite system [for  $\Omega > (D/D_*)E_{Th}$ , when the propagating field does not have enough time to reach the edge of the system], one of the two logarithms entering Eq. (6.47) becomes  $\omega$  independent, cf. Ref. 41. Then  $G_A$  is given by Eq. (6.60), provided that the last exponential factor is replaced by the power-law factor

$$[Z_S^{\text{scr}}]^2 = \left(\frac{\Omega}{\Delta}\right)^{x_z}, \quad (6.62)$$

where

$$x_z = \frac{2}{\pi^2 g} \ln(8\pi v e^2 b). \quad (6.63)$$

### 3. SC island

Now let us consider ZBA effects in the small island geometry shown in Fig. 6. Let us first study the renormalization of the junction transparency  $\gamma$ . Depending on the relation between momentum and the inverse island size  $d^{-1}$ , the geometry of the interface can be either flat, for  $qd \gg 1$ , or point-like, for  $qd \ll 1$ . Then the renormalization of  $\gamma$  can be derived from Eqs. (6.45), (6.52), where the correlator  $\langle [K'(0)]^2 \rangle$  is given by Eq. (6.48), with  $\cos^2(q_x r)$  being formally replaced by  $[1 + \theta(qd - 1)]/2$ . The resulting RG equations can be written in the form (6.56), (6.57) with the function  $r(\zeta)$  defined as

$$r(\zeta) = \frac{1}{4\pi^2 g} \times \begin{cases} 2(\zeta - \zeta_0), & \text{if } \Omega > 2\pi\sigma/d; \\ \zeta_{Th} - \zeta_0, & \text{if } 2\pi\sigma/d > \Omega > \max(\omega_\sigma, \omega_d); \\ \zeta - \zeta_0, & \text{if } \omega_d > \Omega > \omega_\sigma; \\ \zeta_{Th} - \zeta_d, & \text{if } \omega_\sigma > \Omega > \omega_d; \\ \zeta_{Th} - \zeta, & \text{if } \min(\omega_\sigma, \omega_d) > \Omega > E_{Th}; \\ 0, & \text{if } E_{Th} > \Omega. \end{cases} \quad (6.64)$$

The RG equations for  $\gamma(\zeta)$  can be easily solved. Again, we will not present here a general solution depending on the relations between various energy scales,  $E_{Th}(L) = D/L^2$ ,  $\omega_\sigma = 2\pi\sigma/L$ ,  $\omega_d$ ,  $\Omega$ , and  $2\pi\sigma/d$ . Rather we concentrate on the solution in the superconducting region,  $\Omega < \Delta$ , assuming that  $\omega_d \sim \Delta < \omega_\sigma$ . It can be written in the form (6.54) with  $Z_S(\zeta)$  given by

$$Z_S(\zeta) = \exp\left(-\frac{(\zeta - \zeta_\Delta)[\zeta_{Th} - (\zeta + \zeta_\Delta)/2]}{4\pi^2 g}\right) = \exp\left(-\frac{1}{4\pi^2 g} \ln \frac{\Delta}{\Omega} \cdot \ln \frac{\Delta \Omega}{E_{Th}^2(L)}\right). \quad (6.65)$$

The difference by 2 compared to Eq. (6.59) accounts for the difference between spreading over the whole plane and the half-plane.

According to Sec. VI D, in the island geometry,  $G_A$  entering the action as a parameter in Eq. (6.39) gets renormalized after eliminating fast degrees of freedom. In the absence of the ZBA, its renormalization comes from the averaging of two vertices  $S_\gamma$ , cf. Eq. (6.35). With the ZBA effect taken into account, the Andreev action  $S_A$  will renormalize itself similar to the term  $S_\gamma$  in Sec. VI E 2. The corresponding expression reads

$$\langle S_A \rangle = \frac{i\pi}{16} G_A \text{Tr} \langle Q_S(t_1) e^{2iK'_1(t_1)\tau_z} \tilde{Q}(t_1, t_2) \times Q_S(t_2) e^{2iK'_1(t_2)\tau_z} \tilde{Q}(t_2, t_1) \rangle, \quad (6.66)$$

which is written for  $\Omega < \Delta$  when  $Q_S$  given by Eq. (6.5) is local in time. On averaging over fast  $K'_1$ , one finds for the correction to  $G_A$ :

$$\Delta G_A = G_A [e^{-1/2(2K'_1(t_1) - 2K'_1(t_2))^2} - 1]. \quad (6.67)$$

Expanding the exponent and omitting the cross averages, one arrives at

$$\Delta G_A = -4G_A \langle [K'_1(0)]^2 \rangle. \quad (6.68)$$

As a result, we obtain the following modification of the RG equation (6.40):

$$\frac{\partial G_A}{\partial \zeta} = \frac{\mathcal{A}^2 \gamma^2}{4\pi g} - 4r(\zeta)G_A. \quad (6.69)$$

Taking  $\gamma(\zeta)$  from Eq. (6.54) and integrating this differential equation, one obtains the solution for the Andreev conductance:

$$G_A(\Omega) = \frac{G_{T\Delta}^2}{4\pi g} \frac{1 - (\Omega/\Delta)^{2\lambda_g}}{(\lambda_g + \lambda_n) + (\lambda_g - \lambda_n)(\Omega/\Delta)^{2\lambda_g}} \times \exp\left(-\frac{1}{2\pi^2 g} \ln \frac{\Delta}{\Omega} \cdot \ln \frac{\Delta \Omega}{E_{Th}^2}\right) \quad (6.70)$$

where  $G_{T\Delta}$  is the normal-state tunneling conductance at the scale  $\Delta$  renormalized by the ZBA. The coefficient in front of the double logarithm in the ZBA exponent is four times larger than the one for the single electron tunneling, due to the doubled charge of a Cooper pair. Expression (6.70) again shows that the ZBA effect upon  $G_A$  is described by the separate factor  $Z_S^2(\zeta)$  as it should be due to the factorization property discussed above. With the frequency decrease,  $G_A(\Omega)$  first grows logarithmically and then decreases as in the case of the rectangular N-S contact. In the presence of a screening metal gate, the ZBA factor  $[Z_S^{\text{scr}}]^2 = (\Omega/\Delta)^{x_z/2}$  [cf. Eq. (6.62)].

## VII. JOSEPHSON PROXIMITY COUPLING

### A. General treatment and the RG equation

In this section we first rederive semiclassical expressions for the proximity coupling between superconductors, separated by dirty normal metal, and then generalize them, taking into account quantum fluctuations, similar to the way it was done above for the Andreev conductance. The term in the effective action, which is responsible for this coupling, can be written in the form

$$S_J = \frac{1}{2} E_J \text{Tr}(\check{Q}_S^{(1)} \check{Q}_S^{(2)} \sigma_x), \quad (7.1)$$

where superscripts <sup>(1)</sup> and <sup>(2)</sup> refer to two superconductive banks or islands. Using the low-energy representation (6.5) for  $\check{Q}_S$ , and neglecting the Coulomb phase factors  $\exp(iK(t)\tau_z)$ , one rewrites Eq. (7.1) as

$$S_J = -2E_J \int \sin \Theta_1(t) \sin \Theta_2(t) dt, \quad (7.2)$$

with  $\Theta_1$  and  $\Theta_2$  being the classical and quantum components of the phase difference between the superconductors,  $\check{\Theta} = \check{\theta}^{(1)} - \check{\theta}^{(2)}$ . The meaning of the term (7.2) becomes transparent in the initial basis before the Keldysh rotation

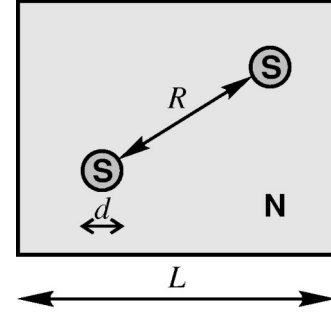


FIG. 9. Two superconductive islands (S) of size  $d$  with separation  $R \gg d$  coupled via a dirty normal film (N) of size  $L$ .

(2.22): it yields the usual expression for the Josephson coupling energy,  $-E_J \cos \Theta$ . Employing Eq. (6.10) one obtains the standard relation between the Josephson current and (the classical component of) the phase difference (with dimensional units restored):

$$I_J = \frac{2eE_J}{\hbar} \sin \Theta_1. \quad (7.3)$$

The term (7.1) can be derived in a way very similar to the derivation of the expression (6.4) in Sec. VI B: one should consider the cross term in the perturbative correction to the action,  $i/2 \langle (S_\gamma^{(1)} + S_\gamma^{(2)})^2 \rangle$ , where  $S_\gamma^{(j)}$  is the boundary action (6.1) at the interface with the superconductor ( $j$ ), and the average is taken over fluctuations of the normal-metal matrix  $Q$ .

The Josephson coupling energy  $E_J$  depends on the geometry of the system, and below we will find  $E_J$  for two geometries shown in Figs. 9 and 10, that are natural counterparts of Figs. 6 and 4 discussed above with respect to the Andreev conductance. In both cases  $E_J$  is given by the Fourier transform [cf. Eqs. (7.6), (7.11)] of the zero-frequency Cooperon amplitude. The latter,  $\mathcal{C}(q, \omega) \equiv \mathcal{J}(\zeta)$ , logarithmically depends on  $\Omega = \max(Dq^2, \omega, T)$ . Within the semiclassical approximation<sup>4</sup> it is given by an integral over the ‘‘center-of-mass’’ Cooperon energy:

$$\mathcal{J}(\zeta) = \frac{G_T^2}{4\nu} \int_0^\Delta \frac{[F(E_+) + F(E_-)] E dE}{(Dq^2)^2 + 4E^2} = \frac{G_T^2}{8\nu} \ln \frac{\Delta}{\Omega}. \quad (7.4)$$

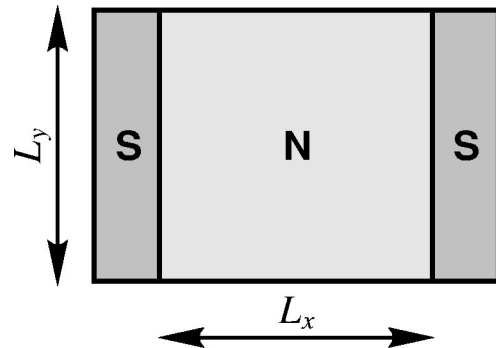


FIG. 10. Two superconductive terminals (S) coupled via a dirty normal film (N) of sizes  $L_x$  and  $L_y$ .

To take into account the effect of interaction in the Cooper channel that makes  $G_T = \mathcal{A}\gamma$  scale dependent, we replace Eq. (7.4) by the following differential RG equation:

$$\frac{\partial \mathcal{J}(\zeta)}{\partial \zeta} = \mathcal{A}^2 \frac{\gamma^2(\zeta)}{8\nu}. \quad (7.5)$$

This RG equation coincides (up to a numerical factor) with the RG equation (6.40) for  $G_A$  in the small island geometry. In the study of the Andreev conductance, such an RG equation appeared as a result of the  $q$  integration and thereby holds only for the geometry of Fig. 6. In the case of the Josephson coupling, Eq. (7.5) comes from the  $E$  integration and thus is insensitive to a particular geometry holding for both setups shown in Figs. 9 and 10.

Although the RG equations (6.40) and (7.5) look similar, the answers for  $G_A$  and  $E_J$  are quite different. The main difference is that  $\mathcal{J}$  is not a final quantity, it should be Fourier transformed to get the Josephson energy  $E_J$ . As a result, only the lowest spacial modes effectively contribute to  $E_J$ , which appears to be exponentially suppressed<sup>4</sup> due to the loss of coherence if temperature or frequency of phase fluctuations exceed the Thouless energy. Note, that in the case shown in Fig. 9, the relevant Thouless energy scale is determined by the interisland distance  $R$ ; we denote this energy as  $\omega_R = D/R^2$  to discern it from  $E_{Th}(L) = D/L^2$  which will also enter results for the ZBA factor for the two-island setup. Below we will assume that both  $T$  and  $\omega$  are much lower than  $E_{Th}$  for the rectangular geometry, and much lower than  $\omega_R$  for the island geometry.

Equation (7.5) is sufficient to find the proximity coupling energy in the absence of the zero-bias anomaly effects (which will be incorporated in the next subsection). We start from an example of two small SC islands of size  $d$ , separated by the distance  $R \gg d$  (see Fig. 9). In this case

$$E_J(R) = \int \frac{d\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{R}} \mathcal{C}(q) = \frac{1}{2\pi} \int_0^\infty q dq J_0(qR) \mathcal{J}(\zeta_q), \quad (7.6)$$

where  $\zeta_q = \ln(1/Dq^2\tau)$ , and the use of the continuous Fourier transform implies that the distance between islands is much shorter than the total size of the film,  $L \gg R$ . Using the identity  $J_0(x) = (1/x)(\partial/\partial x)[xJ_1(x)]$ , and integrating by parts we obtain

$$E_J(R) = \frac{1}{\pi R} \int_0^\infty dq J_1(qR) \frac{\partial \mathcal{J}(\zeta_q)}{\partial \zeta_q}. \quad (7.7)$$

Since  $\mathcal{J}(\zeta_q)$  is a very slow function of  $q$  (provided that  $g \gg 1$ ), the  $q$  integration converges rapidly near  $q \sim R^{-1}$ . Then we may take  $\mathcal{J}$  at  $\zeta_q = \ln(R^2/D\tau)$  and perform the remaining integration that yields

$$E_J(R) = \frac{1}{\pi R^2} \left. \frac{\partial \mathcal{J}(\zeta)}{\partial \zeta} \right|_{\zeta = \ln(R^2/D\tau)}. \quad (7.8)$$

Combining Eqs. (7.8) and (7.5) with Eq. (6.31), we obtain

$$E_J(R) = \frac{G_T^2}{8\pi\nu R^2} \frac{4(\xi_\Delta/R)^{4\lambda_g}}{[(1+\lambda_n/\lambda_g) + (1-\lambda_n/\lambda_g)(\xi_\Delta/R)^{4\lambda_g}]^2}. \quad (7.9)$$

Thus, the effect of repulsive interaction in the Cooper channel is to produce an anomalous power-law exponent  $x_J = 2 + 4\lambda_g$ , describing decay of the Josephson coupling at long distances:  $E_J \sim R^{-x_J}$ .

Now we proceed to the case of the rectangular geometry shown in Fig. 10. In this case the set of eigenfunctions with the proper boundary conditions reads [cf. Eq. (6.16)]

$$\psi_{mn}(x,y) = \frac{2 \cos q_m x \cos k_n y}{\sqrt{(1+\delta_{m,0})(1+\delta_{n,0})L_x L_y}}, \quad q_m = \frac{\pi}{L_x} m, \\ k_n = \frac{\pi}{L_y} n, \quad m, n = 0, 1, \dots \quad (7.10)$$

The difference with the case of two islands is that now one has to sum over the single component  $q_m$  of the wave vector only (cf. Sec. VI C):

$$E_J(L_x, L_y) = \frac{1}{L_x L_y} \sum_{m=-\infty}^{\infty} (-1)^m \mathcal{C}(q_m). \quad (7.11)$$

Equation (7.5) for  $\mathcal{J} = \text{const} \cdot G_A$  has been already solved in Sec. VI D; substituting the solution into Eq. (7.11) and using the equality  $\sum_{m=-\infty}^{\infty} (-1)^m = 0$  we transform  $E_J$  to the form

$$E_J(L_x, L_y) = \frac{G_T^2}{4\nu} \frac{1}{L_x L_y} \frac{\lambda_g}{\lambda_g - \lambda_n} \\ \times \sum_{m=-\infty}^{\infty} \frac{(-1)^m}{(\lambda_g + \lambda_n) + (\lambda_g - \lambda_n)(E_{Th} m^2 / \Delta)^{2\lambda_g}}, \quad (7.12)$$

where  $E_{Th} = D/L_x^2$ . Consider this sum and write it as  $\sum_{m=-\infty}^{\infty} (-1)^m f_m$ . For  $\lambda_g \ll 1$ ,  $f_m$  is a very slow function for  $m \neq 0$  but has a cusp at  $m=0$ . Then the sum can be well estimated as  $f_0 - f_1$  and is equal to

$$\frac{(\lambda_g - \lambda_n)(E_{Th}/\Delta)^{2\lambda_g}}{(\lambda_g + \lambda_n)[(\lambda_g + \lambda_n) + (\lambda_g - \lambda_n)(E_{Th}/\Delta)^{2\lambda_g}]}. \quad (7.13)$$

As a result, we obtain

$$E_J(L_x, L_y) = \frac{G_T^2}{8\nu(\lambda_g + \lambda_n)L_x L_y} \\ \times \frac{2(\xi_\Delta/L_x)^{4\lambda_g}}{(1+\lambda_n/\lambda_g) + (1-\lambda_n/\lambda_g)(\xi_\Delta/L_x)^{4\lambda_g}}. \quad (7.14)$$

In the above expression  $G_T$  is the total normal-state tunneling conductance proportional to the width  $L_y$  of the contact. In the rectangular geometry a nontrivial exponent  $x'_J = 1 + 4\lambda_g$  enters the  $E_J(L_x)$  dependence:  $E_J \propto L_x^{-x'_J}$ . In the case of weak disorder,  $\lambda_g \ln(L_x/\xi_\Delta) \ll 1$ , the result (7.14) reduces to the one obtained in Ref. 4, whereas Eq. (7.9) reduces to

the form used in Ref. 44 [an extra factor  $4/\pi^2$  in Eq. (18) of Ref. 44 is due to the difference in the definition of the dimensionless tunneling conductance  $G_T$ ]. Both of the above results (7.9), (7.14) were obtained neglecting the zero-bias anomaly effects. Now we proceed to take them into account.

### B. Proximity coupling in the presence of the zero-bias anomaly

As it was explained in Sec. VI E 1, the effect of the Coulomb phase  $\tilde{K}(t)$  fluctuations (which are due to the long-range Coulomb potential) can be separated (factorized) from the effect of the interaction in the Cooper channel. Below we will use this factorization explicitly.

Consider first the two island geometry of Fig. 9. Using the gauge-transformed form (6.42) of the boundary action term, and repeating the steps that lead to Eqs. (7.5) and (7.6), we obtain now the Josephson coupling energy  $E_J(R)$  as a time-domain convolution of the Cooperon and the ZBA exponent:

$$E_J(R) = \int dt Z_J^2(t, R) \int \frac{d\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q} \cdot \mathbf{R}} \mathcal{C}(q, t), \quad (7.15)$$

where  $C(q, t) = \int (d\omega/2\pi) \mathcal{J}(\zeta) e^{-i\omega t}$  is the time-domain Cooperon amplitude, and  $\zeta = \ln 1/\Omega\tau$  with  $\Omega = \max(Dq^2, \omega)$ . The ZBA factor  $Z_J(t, R)$  is defined as [cf. Eq. (6.47)]

$$\begin{aligned} -\ln Z_J(t, R) &= \langle [K_1'(t, R) - K_1'(0, 0)]^2 \rangle \\ &= \frac{1}{\pi^2 g} \int_0^\Delta \frac{d\omega}{\omega} \int_{\max(\omega/2\pi\sigma, 1/L)}^{\sqrt{\omega/D}} \frac{dq}{q} \\ &\quad \times [1 - \cos \omega t J_0(qR)], \end{aligned} \quad (7.16)$$

where  $L \gg R$  is the size of the metal film. Here we assume that  $\omega_d \sim \Delta$ , and include all ZBA fluctuations with frequency  $\omega \gg \Delta$  into redefinition of the normal-state tunneling conductance  $G_{T\Delta}$  at the scale  $\Delta$ . In the absence of the ZBA,  $Z_J = 1$  and Eq. (7.15) reduces to Eq. (7.6), with  $\mathcal{C}(q) \equiv \mathcal{C}(q, \omega = 0)$ . The lower limit of the  $q$  integration depends on the relation between the field spreading distance,  $2\pi\sigma/\omega$ , and the sheet size,  $1/L$ . To be more specific, we will assume, following Sec. VI E, that  $\Delta/2\pi\sigma \ll L$ . Then the ZBA exponent is equal to

$$Z_J(t, R) = \exp\left(-\frac{1}{4\pi^2 g} \ln \frac{\Delta}{\omega_R} \cdot \ln \frac{\Delta \omega_R}{E_{Th}^2(L)}\right), \quad (7.17)$$

where  $\omega_R = D/R^2$ ,  $E_{Th}(L) = D/L^2$ , and it is assumed that  $t \sim \omega_R^{-1}$ . Now substituting  $\mathcal{J}(\zeta)$  from Eq. (7.5) into Eq. (7.15), using the trick which led to Eq. (7.7) and the identity  $\partial \mathcal{J}(\zeta)/\partial \zeta_q = \theta(Dq^2 - |\omega|) \partial \mathcal{J}(\zeta_q)/\partial \zeta_q$ , and performing a trivial integration over  $\omega$ , we obtain

$$E_J(R) = \frac{1}{\pi^2 R} \int dt Z_J^2(t, R) \int_0^\infty dq J_1(qR) \frac{\partial \mathcal{J}(\zeta_q)}{\partial \zeta_q} \frac{\sin Dq^2 t}{t}. \quad (7.18)$$

Both momentum and time integrals converge fairly well in the vicinity of  $q \sim R^{-1}$  and  $t \sim \omega_R^{-1}$ , respectively. As a result, to logarithmic accuracy we get

$$E_J(R) = \frac{1}{\pi R^2} Z_J^2(\zeta) \frac{\partial \mathcal{J}(\zeta)}{\partial \zeta} \Bigg|_{\zeta = \ln(R^2/D\tau)}. \quad (7.19)$$

Note that the same result could be obtained within the RG approach with the help of the equation similar to Eq. (6.69):

$$\frac{\partial \mathcal{J}}{\partial \zeta} = \frac{A^2 \gamma^2}{8\nu} - 4r(\zeta) \mathcal{J} \quad (7.20)$$

with  $r(\zeta) = -(1/2) \partial \ln Z_J / \partial \zeta$ .

Finally, we obtain the Josephson coupling energy for two islands:

$$\begin{aligned} E_J(R) &= \frac{G_{T\Delta}^2}{8\pi\nu R^2} \frac{4(\xi_\Delta/R)^{4\lambda_g}}{[(1 + \lambda_n/\lambda_g) + (1 - \lambda_n/\lambda_g)(\xi_\Delta/R)^{4\lambda_g}]^2} \\ &\quad \times \exp\left(-\frac{2}{\pi^2 g} \ln \frac{R}{\xi_\Delta} \cdot \ln \frac{L^2}{\xi_\Delta R}\right). \end{aligned} \quad (7.21)$$

The presence of the total size  $L$  of the film in the above expression for the interisland proximity coupling may appear to be unexpected. It originates from the fact that electric field propagates much faster than electron density: during diffusion time  $R^2/D$  corresponding to the interisland distance, the electric field propagates the distance  $L(R) = \kappa_2 R^2$ , where  $\kappa_2 = 4\pi\nu e^2$  is the inverse 2D screening length. The effective electric impedance that determines the ZBA factor does depend on the distance  $L(R)$  as long as it is shorter than the total size  $L$ . If the dirty film is of metallic origin,  $\kappa_2^{-1}$  is of the order of Angströms, so  $L(R)$  easily exceeds any reasonable size of the film even for  $R$  in the submicron range. In this case the effective impedance is determined by the film size  $L$ . All these considerations do not hold for the case of the screened Coulomb potential (4.24); in that case the exponential factor in Eq. (7.21) should be replaced by  $(\xi_\Delta/R)^{x_\zeta}$ , cf. the end of Sec. VI.

Consider now the ZBA effect for the rectangular geometry (see Fig. 10). In this case one has

$$E_J(L_x, L_y) = \frac{1}{L_x L_y} \int dt Z_J^2(t, L_x) \sum_{m=-\infty}^{\infty} (-1)^m \mathcal{C}(q_m, t), \quad (7.22)$$

cf. Eqs. (7.11) and (7.15). Again, the relevant scale for the  $t$  integration is given by  $E_{Th}^{-1} = R^2/D$ . Then one can integrate over  $t$  keeping  $Z_J(t) = Z_J(E_{Th}^{-1})$  constant that amounts to multiplying the previous result (7.14) by  $Z_J^2(E_{Th}^{-1})$ , provided that  $G_T$  is substituted by  $G_{T\Delta}$ . The ZBA factor for a flat N-S interface is given by

$$Z_J(E_{Th}^{-1}, L_x) = \exp\left(-\frac{1}{2\pi^2 g} \ln^2 \frac{\Delta}{E_{Th}}\right), \quad (7.23)$$

where  $E_{Th} = D/L_x^2$ . Therefore, we obtain



$$\begin{aligned}
E_J(L_x, L_y) &= \frac{G_{T\Delta}^2}{8\nu(\lambda_g + \lambda_n)L_x L_y} \\
&\times \frac{2(\xi_\Delta/L_x)^{4\lambda_g}}{(1 + \lambda_n/\lambda_g) + (1 - \lambda_n/\lambda_g)(\xi_\Delta/L_x)^{4\lambda_g}} \\
&\times \exp\left(-\frac{4}{\pi^2 g} \ln^2 \frac{L_x}{\xi_\Delta}\right). \quad (7.24)
\end{aligned}$$

For the screened Coulomb potential, the last factor in Eq. (7.24) is replaced by  $(\xi_\Delta/L_x)^{2x_z}$ . Note, finally, that both expressions (7.21) and (7.24) refer to the low-temperature limit  $T \ll \hbar D/R^2, \hbar D/L_x^2$ .

### VIII. DISCUSSION

We have developed field theory functional formalism of the Keldysh type for disordered superconductors. This approach provides a regular method for the treatment of all kinds of quantum fluctuations beyond the standard semiclassical theory of superconductivity<sup>18–20</sup> which can be understood as the saddle-point approximation of our theory. The theory is formulated in terms of the local (in space) matrix order parameter whose components correspond to the retarded, advanced and Keldysh Green's functions. The main advantage of our approach with respect to the standard Matsubara replica formalism<sup>12</sup> is that it provides the possibility of treating nonequilibrium problems.

A general formulation of the theory involves  $8 \times 8$  matrices  $Q_{i,i'}$ , composed as the direct product of the  $2 \times 2$  blocks corresponding to the Nambu, time-reversal, and Keldysh spaces. However, in order to present our approach in the most transparent form, we restricted here our specific calculations to the case of spin-independent interactions, and neglected the usual weak-localization conductivity corrections, which makes it possible to use reduced  $4 \times 4$  matrices. We focused here on the fluctuational effects specific for the dirty superconductive films, whose relative magnitude is known<sup>11,12</sup> to be of the order of  $(1/g)\ln^2(L/l)$  (in the two-dimensional limit), and thus can be considered independently of the normal-metal weak-localization and interaction corrections<sup>6,7</sup> which are  $\propto (1/g)\ln(L/l)$ . There are two physically different types of these effects: (i) the Coulomb-induced repulsive contribution to the electron-electron interaction in the Cooper channel (coming from the ‘‘diffusive’’ frequency region  $\omega \ll Dq^2$ ), and (ii) reduction of the averaged single-particle density of states due to high-frequency ( $\omega \gg Dq^2$ ) fluctuations of electric potential. We have shown, following the approach developed in Ref. 22, that these two effects can be separated at the nonperturbative level, by ‘‘gauging away’’ phase factors induced by the fluctuating long-range electric potential. The first effect was treated by the renormalization group method within the single-loop approximation. Being applied to the case of uniformly disordered superconductive films, this method essentially reproduces the well-known results by Finkelstein<sup>11–13</sup> for the Coulomb-induced suppression of the superconductive transition temperature  $T_c$ . The second effect is formally similar to the ‘‘infrared catastrophe’’ of quantum electrodynamics; in our technique it is taken into account in an essentially exact

way, similar to the normal-metal case treated in Ref. 22. This second effect (analogous to the zero-bias anomaly of the tunneling conductance), contrary to the first one, does depend on the long-range part of the bare Coulomb potential, and, thus, can be modified by the presence of an additional external screening (provided, e.g., by a metallic gate nearby the dirty film).

It was argued previously (cf. Ref. 13) that the fluctuations responsible for the zero-bias anomaly effect are specific for the single-particle density of states, due to the gauge noninvariance of the single-particle Green's function, whereas the same fluctuations do not contribute to other (gauge-invariant) physical quantities like  $T_c$ . We have calculated two important characteristics of mesoscopic S/N structures, the Andreev subgap conductance and the Josephson proximity coupling, and found that they are affected by the zero-biased anomaly effect as well as by the Finkelstein effect. To be more specific, we have calculated linear Andreev conductance  $G_A(\Omega)$  in the presence of the tunneling barrier (N-I-S structure), for two different geometries: (i) a small SC island ‘‘sitting’’ on a large-area dirty metal (Fig. 6), and (ii) a rectangular normal film between superconductive and normal reservoirs (Fig. 4). Our results for those two cases are given by Eqs. (6.70) and (6.60) correspondingly. In these formulas  $\Omega$  stands for the infrared cutoff of the renormalization procedure, i.e.,  $\Omega = \max(T, \omega, \tau_\phi, D/L^2)$ . A detailed generalization of these results for the steady-state nonlinear Andreev current  $I_A(V)$  can be obtained with the use of formulas from Appendix B; qualitatively the behavior of  $G_A(V) = dI_A/dV$  can be found by substituting  $2eV$  for  $\Omega$  in Eqs. (6.70) and (6.60).

The second new physical quantity we have studied is the Josephson proximity coupling  $E_J$  as a function of the size of the normal region between two SC contacts. It was calculated, in the low-temperature limit  $T \ll E_{Th}$ , for the case of two small islands in contact with a dirty normal metal (Fig. 9), as well as for the rectangular film between two SC banks (Fig. 10). The electron-electron repulsion in the Cooper channel leads to the appearance of an anomalous scaling for  $E_J$ . In particular, for two small SC islands  $E_J(R) \propto R^{-2-2/\pi\sqrt{g}}$ , whereas for the rectangular contact the Josephson energy decays with the film length as  $E_J(L_x) \propto L_x^{-1-2/\pi\sqrt{g}}$ . In addition, the ‘‘ZBA effect’’ adds to the above behavior a log-normal suppression factor  $\exp[-(4/\pi^2 g)\ln^2(L_x/\xi_\Delta)]$  for the rectangular contact, and a similar factor for the island geometry. Full results for both geometries are given, at the low-temperature limit  $T \rightarrow 0$ , by Eqs. (7.21) and (7.24). Screening of the long-range Coulomb interaction in the film by an external gate makes the ZBA suppression weaker, it reduces then to the power-law factor  $(\xi_\Delta/L_x)^{2x_z}$  and  $(\xi_\Delta/R)^{x_z}$  for the rectangular and island geometries correspondingly, where  $x_z$  is defined in Eq. (6.63).

In the present calculations we did not take into account the existence of a finite decoherence time  $\tau_\phi$  in normal metals at  $T > 0$ . Qualitatively, the effect of nonzero  $\tau_\phi^{-1}$  is to suppress the central peak in  $I(V)$  predicted within the semiclassical theory, i.e., it acts in the way similar to the effect of quantum corrections we have studied in this paper (we emphasize that these quantum effects have nothing to do with ‘‘zero-temperature decoherence’’). That is why the decoher-

ence rate  $\tau_\phi^{-1}$  extracted from the data on the Andreev conductance might be overestimated unless quantum fluctuations are taken into account in the data analysis. Another unsolved theoretical problem is to go beyond the lowest-order expansion over the tunneling conductance  $G_T$ , that becomes necessary if  $G_T$  approaches or exceeds the conductance of the metal region.

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### APPENDIX A: ALTERNATIVE PROOF OF EQ. (6.12)

In this Appendix we present an alternative proof of Eq. (6.12) with the help of the fluctuation-dissipation theorem. Expanding the action (6.8) to the second order in the phase variables, we obtain

$$S_A^{(2)}[\vec{\theta}] = \frac{i}{4} G_A \int \frac{d\omega}{2\pi} \vec{\theta}^T(-\omega) \omega \Pi_\omega^{-1} \vec{\theta}(\omega). \quad (\text{A1})$$

Hence, in the Gaussian approximation, the correlator of phases is given by

$$\langle \theta_i(\omega) \theta_j(-\omega) \rangle = \frac{2(\Pi_\omega)^{ij}}{G_A \omega}. \quad (\text{A2})$$

Employing Eq. (6.9) relating the phase of the island with the applied voltage, we get for the spectral voltage correlator:

$$\langle V_\omega V_{-\omega} \rangle = \frac{1}{e^2 G_A} \omega \coth \frac{\omega}{2T}, \quad (\text{A3})$$

that, according to the Nyquist formula, results in Eq. (6.12).

### APPENDIX B: NONLINEAR ANDREEV CONDUCTANCE

To derive a nonlinear Andreev current  $I_A(V)$  we start from Eq. (6.21). Tracing over the Nambu space we reduce the expression for the action to the form similar to Eq. (6.8), with  $G_A$  being replaced by  $G_A(\omega)$ . Calculating the current following Eq. (6.10), we find

$$\begin{aligned} I_A(t) &= \frac{e}{4} \int dt_1 \int \frac{d\omega}{2\pi} \omega G_A(\omega) e^{-i\omega(t_1-t)} \\ &\times \left\{ \left\langle \left( e^{i\theta_1(t_1)-i\theta_1(t)} - e^{-i\theta_1(t_1)+i\theta_1(t)} \right) \cos\theta_2(t_1) \cos\theta_2(t) \right\rangle \right. \\ &\quad - 2i \coth \frac{\omega}{2T} \left[ \langle e^{i\theta_1(t_1)} \cos\theta_1(t) \sin\theta_2(t_1) e^{-i\theta_1(t)} \rangle \right. \\ &\quad \left. \left. + \langle e^{-i\theta_1(t_1)} \cos\theta_1(t) \sin\theta_2(t_1) e^{i\theta_1(t)} \rangle \right] \right\}. \quad (\text{B1}) \end{aligned}$$

Angular brackets in Eq. (B1) mean averaging over quantum dynamics of the phase  $\theta(t)$ . In the case when fluctuations of  $\theta$  can be neglected, the second line in Eq. (B1) should be omitted, whereas in the first line we set  $e^{i\theta_1(t_1)-i\theta_1(t)} = e^{2ieV(t_1-t)}$ . As a result, integration over  $t_1$  produces  $2\pi\delta(\omega - 2eV)$ , and one obtains

$$I_A = e^2 V G_A(2eV). \quad (\text{B2})$$

The function  $G(\omega)$  in Eq. (B2) should be determined with the account of fluctuational corrections due to the interaction in the normal metal, as discussed in Secs. VI C and VI D. The effect of the zero-bias anomaly also can be taken into account along the same lines, cf. Sec. VI E. However, the RG method employed there does not determine the dependence of the ZBA factor  $Z_S(\Omega)$  [cf. Eq. (6.55)] upon the ratio  $2eV/T$ . To find this dependence, we use the factorization property discussed in Sec. VI E 1 and note that the low-energy ZBA effect can be accounted for by the replacement  $\vec{\theta}(t) \rightarrow \vec{\theta}_{\text{eff}}(t) = \vec{\theta}(t) - 2\vec{K}(t)$ , cf. discussion after Eq. (6.51). After such a replacement the correlation functions entering Eq. (B4) can be expressed via the correlation matrix

$$\mathcal{D}(\omega) = \begin{pmatrix} \mathcal{D}^K(\omega) & \mathcal{D}^R(\omega) \\ \mathcal{D}^A(\omega) & 0 \end{pmatrix}, \quad (\text{B3})$$

where, similar to Eqs. (4.29) and (4.30), the matrix elements  $\mathcal{D}_{ij}(\omega) = -2i \langle \Psi_i(\omega) \Psi_j^\dagger(-\omega) \rangle$ , and  $\vec{\Psi}(t) = e^{-2i\vec{K}(t)}$ . As a result, the whole expression for the current can be written as

$$\begin{aligned} I_A(V) &= \frac{e}{4} \int \frac{d\omega}{2\pi} \omega G_A(\omega) \left[ i \mathcal{D}^K(2eV - \omega) \right. \\ &\quad \left. + 2 \coth \frac{\omega}{2T} \text{Im} \mathcal{D}^R(2eV - \omega) \right]. \quad (\text{B4}) \end{aligned}$$

Note that  $i\mathcal{D}^K(\omega)$  is a real positive function. Below it will be convenient to rewrite Eq. (B4) via the forward and backward correlation functions  $\mathcal{D}^{>(<)}(\omega)$  defined as

$$\mathcal{D}^K = \mathcal{D}^> + \mathcal{D}^<, \quad \mathcal{D}^R - \mathcal{D}^A = \mathcal{D}^> - \mathcal{D}^<. \quad (\text{B5})$$

In the equilibrium state, the detailed balance relation  $\mathcal{D}^<(\omega) = e^{-\omega/T} \mathcal{D}^>(\omega)$  is valid for bosonic correlation functions, in addition to the general relations (B5). In terms of the original [before the Keldysh rotation Eq. (2.22)] bosonic variables,  $\mathcal{D}^>(t) = -i \langle \Psi(t) \Psi^\dagger(0) \rangle$ . The correlators labeled by the index ‘‘>’’ contain fields taken at the forward branch of the Keldysh contour  $\mathcal{C}$  (for the sake of brevity we will omit the corresponding index ‘‘1’’).

If the correction to  $G_A(\omega)$  from the interaction vertex  $\lambda$  in the Cooper channel can be neglected, it can be written as

$$G_A(\omega) = \frac{G_T^2}{2\nu} \int \frac{dE}{\omega} \left[ \tanh \frac{E_+}{2T} - \tanh \frac{E_-}{2T} \right] C(E), \quad (\text{B6})$$

where  $E_{\pm} = E \pm \omega/2$ , and

$$C(E) = \frac{1}{L_x L_y} \text{Re} \sum_{\mathbf{q}} \frac{1}{-2iE + Dq^2} \quad (\text{B7})$$

is the Cooperon amplitude. The sum in Eq. (B7) goes over Cooperon eigenmodes (cf. derivation in Sec. VI C 1).

Combining Eqs. (B4)–(B7), we obtain finally

$$I_A(V) = i \frac{eG_T^2}{4\nu} \int_{-\infty}^{\infty} C(E) dE \int_{-\infty}^{\infty} \frac{dE'}{2\pi} \left[ \tanh \frac{E - E'/2 + eV}{2T} - \tanh \frac{E + E'/2 - eV}{2T} \right] \mathcal{D}^>(E') \frac{1 - e^{-2eV/T}}{1 - e^{(E' - 2eV)/T}}. \quad (\text{B8})$$

The result (B8) is equivalent to the one obtained in Ref. 28 by the method of analytic continuation. In the  $T \rightarrow 0$  limit expression (B8) simplifies and can be transformed to the following form for the differential conductance:

$$\frac{dI_A}{dV} = \frac{e^2 G_T^2}{\nu} \int_{-\infty}^{2eV} dE \mathcal{P}(E) C(eV - E/2), \quad (\text{B9})$$

where  $\mathcal{P}(E) = (i/2\pi) \mathcal{D}^>(E)$ .

Note, finally, that the results (B8), (B9) are valid in the situation when the order parameter phase  $\theta(t)$  is subject to quantum fluctuations as well; in this case the correlation function  $\mathcal{D}^>$  is defined as  $\mathcal{D}^>(t) = -i \langle e^{i\tilde{\theta}(t)} \Psi(t) e^{-i\tilde{\theta}(0)} \Psi^+(0) \rangle$ , where  $\tilde{\theta}(t) = \theta(t) - 2eVt$ .

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