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Nuclear Physics B 506 [FS] (1997) 665–684

NUCLEAR PHYSICS B

Supersymmetric model of a 2D long-range Bose liquid

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Received 26 March 1997; revised 31 July 1997; accepted 4 September 1997

Abstract

The model Hamiltonian of a two-dimensional Bose liquid (proposed earlier by Kane et al. as the Hamiltonian which has Jastrow-type wavefunctions $\Psi_0\{r_i\} \propto \prod_{j>k} |r_j - r_k|^{2\alpha}$ as the ground-state solution), is shown to possess non-relativistic supersymmetry. For the special value of the coupling constant $\alpha = 1/2$ the quantum mechanics described by this Hamiltonian is shown to be equivalent to the dynamics of (complex) eigenvalues of a random Gaussian ensemble of normal complex matrices. For general α, an exact relation between the equal-time current–current and density–density correlation functions is obtained, and used to derive an asymptotically exact (at low wavevectors q) spectrum of single-particle excitations beyond the superfluid ground state (realized at low α's). The ground state Ψ_0 at very large α is shown to be of "quantum hexatic" type, possessing long-range orientational order and quasi-long-range translational order *but with zero shear modulus*. Possible scenario's of the ground-state phase transitions as a function of α are discussed. © 1997 Elsevier Science B.V.

PACS: 05.30.Jp; 71.10.Hf; 12.60.Jv
Keywords: Superfluidity; Non-relativistic supersymmetry; Matrix models

1. Introduction

It is commonly believed that a system of Bose particles at zero temperature may exist only in one of two possible ground states: superfluid (SF) or crystalline (CR). Most probably this is indeed the case if the interaction between bosons is short-range (compared to the mean interparticle distance $n^{-1/2}$). However, the situation is much

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PII S0550-3213(97)00616-0

less clear if long-range and/or long-time interactions are involved. Two physical situations of the problem of a two-dimensional (2D) Bose system interacting with a 2D Maxwell gauge field were considered in [1]: classical thermodynamics of vortex line liquid in high-temperature superconductors (see Ref. [2] for an extensive review) and low-temperature quantum properties of a liquid of holons (arising in the RVB-type theories [3] of a doped Mott dielectric [4]). Arguments were given [1] in favor of the existence of a new unusual ground state which is still *liquid*, but is *not superfluid*. More recently the same problem was attacked using another approach [5], also in relation with RVB-type theories. Still the theoretical arguments in favor of the existence of such an unusual ground state are not quite conclusive; the results of different computer simulations are in mutual disagreement [6,7]. Whereas in [6] two phase transitions and an intermediate state were observed, in [7] a direct first-order transition from SF to CR state was found. Note, however, that the model Hamiltonians simulated in these two numerical studies were different in that the retarded interactions were presented only in [6].

There are two different questions related with this problem: (i) can one invent such a ground state in a self-consistent way? and (ii) is it realized in some model with a physically realistic long-range interaction? In the present paper we try to approach the answer to the first of the above questions, using a special model of a 2D Bose liquid, introduced originally by Kane, Kivelson, Lee and Zhang (KKLZ) [8]. The KKLZ model Hamiltonian is constructed in such a way that its exact ground-state wavefunction is of the Jastrow type, i.e. it is a product of 1-particle and 2-particle dependent factors. With the special choice of the Jastrow function the main (in the long-wavelength limit) term of the KKLZ Hamiltonian is given by a pair-wise logarithmic interaction, i.e. it resembles a 2D static Coulomb potential of the real problem discussed in [1]. On the other hand, a special symmetry (in fact, a kind of a supersymmetry) of the KKLZ Hamiltonian makes it possible to obtain a number of exact relations between the equal-time $T = 0$ correlation functions, which help to analyze the general structure of the theory.

The paper is organized as follows: in Section 2 we derive the KKLZ Hamiltonian from the condition that it is given by the zero-fermion sector of a supersymmetric non-relativistic parity-conserving theory, which resembles (in the long-wavelength limit) the system of 2D bosons with a logarithmic interaction. The coupling constant α of our theory is proportional to the charge of 2D bosons, thus the ground state possesses superfluidity at low α. In Section 3 we analyze a special case ($\alpha = 1/2$) of the obtained model and show that it is equivalent to the quantum mechanics of a Gaussian matrix model with normal complex matrices (in the same sense as quantum mechanics of a Gaussian Hermitian matrix model is equivalent to 1D free fermions [9]). In Section 4 we are back to the general case and derive an exact real-space relation between the equal-time current–current and density–density correlation functions (extending similar results found in [10] for the Calogero–Sutherland model to a 2D case). These relations are then used in Section 5 to derive a relation between momentum-space transverse current–current correlation function and the static structure factor, and in Section 6 to find an

exact form of the low-momentum excitation spectrum $\omega(q)$ (which appears to be given by the one-loop correction to the mean-field approximation). Section 7 is devoted to the discussion of the "crystalline" ground state of the KKLZ Hamiltonian; we show that such a state has in fact zero shear modulus and thus should be considered as a kind of "quantum hexatic" (QHX), in some analogy with the classical hexatic known to exist as an intermediate phase within the Halperin–Nelson picture of thermal 2D melting [11]. In Section 8 we speculate on possible properties of the additional ground state which may be sandwiched between the SF and QHX states. Our conclusions are presented in Section 9. Appendix A is devoted to the description a model of classical stochastic dynamics which is equivalent to our quantum-mechanical problem; we show how to use this equivalence for efficient numerical simulations of some quantum-mechanical expectation values. Finally, in Appendix B we derive the low-frequency excitation spectrum in the QHX state.

2. The KKLZ Hamiltonian: SUSY structure and general framework

We will see now that the KKLZ Hamiltonian can be considered as an example of a many-body generalization of the supersymmetric quantum mechanics introduced in Refs. [12,13]. We define a pair of the conjugated "supercharge" operators Q and Q^+:

$$Q = \sum_{j,\alpha} q_{j,\alpha} a^+_{j,\alpha}, \qquad Q^+ = \sum_{j,\alpha} q^+_{j,\alpha} a_{j,\alpha}, \tag{1}$$

where fermionic operators $a_{j,\alpha}$ and $a^+_{j,\alpha}$ correspond to each component of the jth particle's coordinate vector $r_{j,\alpha}$ (with $\alpha = 1,2$), and

$$q_{j,\alpha} = -i\frac{\partial}{\partial r_{j,\alpha}} + i \cdot 2\alpha\left(\sum_{k\neq j} \frac{\partial \ln|\boldsymbol{r}_j - \boldsymbol{r}_k|}{\partial r_{j,\alpha}} - \pi n r_{j,\alpha}\right),$$
$$q^+_{j,\alpha} = -i\frac{\partial}{\partial r_{j,\alpha}} - i \cdot 2\alpha\left(\sum_{k\neq j} \frac{\partial \ln|\boldsymbol{r}_j - \boldsymbol{r}_k|}{\partial r_{j,\alpha}} - \pi n r_{j,\alpha}\right). \tag{2}$$

Note that the introduction of two fermionic operators a_{j1}, a_{j2} is necessary in order to construct a P-invariant theory; if the condition of P-invariance were relaxed, the use of just one (per particle) fermionic operator c_j corresponding to the holomorphic coordinate $z_j = x_j + iy_j$ would be possible. Now we define the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}\{Q, Q^+\} \tag{3}$$

and note that this Hamiltonian, when acting on the state of "Fermi vacuum" (defined by $a_{j,\alpha}|0_F\rangle = 0$), $H = \mathcal{H}|0_F\rangle$, is equivalent to the Kane–Kivelson–Lee–Zhang (KKLZ) Hamiltonian [8] (see also Ref. [1])

$$H - E^0 = H_{\mathrm{KKLZ}} = \frac{1}{2m}\sum_i \Big\{ -\nabla_i^2 + 2\alpha \sum_{j\neq i} \nabla_i^2 \ln|\boldsymbol{r}_i - \boldsymbol{r}_j| + 4\alpha^2 \sum_{j\neq i, k\neq i} \nabla_i \ln|\boldsymbol{r}_i - \boldsymbol{r}_j| \nabla_i \ln|\boldsymbol{r}_i - \boldsymbol{r}_k| + (2\pi\alpha n)^2 \boldsymbol{r}_i^2 \Big\}, \tag{4}$$

where $E^0 = -2\pi\alpha n N[1 + \alpha(N-1)]/m$. The corresponding ground-state wavefunction is given by

$$\Psi_0(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \mathrm{const} \cdot \prod_{j>k} |\boldsymbol{r}_j - \boldsymbol{r}_k|^{2\alpha} \exp\left(-\pi\alpha n \sum_i r_i^2 \right). \tag{5}$$

Alternatively, the Hamiltonian can be written directly in terms of the $q_{j,\alpha}$ operators:

$$H = \frac{1}{2m} \sum_j q_{j,\alpha}^+ q_{j,\alpha}. \tag{6}$$

Each of the $q_{j,\alpha}$ operators annihilates the ground state (5), so the ground-state energy corresponding to H is zero. The Hamiltonian (4) contains two-body and three-body interactions; in terms of the local particle's density $\rho(\boldsymbol{r})$ the corresponding potential energy is given by

$$U = \frac{1}{2}\int d^2r_1\, d^2r_2\, \rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2) V_2(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \int d^2r_1\, d^2r_2\, d^2r_3\, \rho(\boldsymbol{r}_1)\rho(\boldsymbol{r}_2)\rho(\boldsymbol{r}_3) V_3(\boldsymbol{r}_1 - \boldsymbol{r}_2, \boldsymbol{r}_1 - \boldsymbol{r}_3), \tag{7}$$

where

$$V_2(\boldsymbol{r}) = \frac{4\pi\alpha}{m}\delta(\boldsymbol{r}) + \frac{4\alpha^2}{m}\frac{1}{r^2}, \qquad V_3(\boldsymbol{r}, \boldsymbol{r}') = \frac{2\alpha^2}{m}\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2 r'^2}.$$

Writing $\rho(\boldsymbol{r}_1) = n + \delta\rho(\boldsymbol{r}_1)$, where n is the mean density, and integrating over $\boldsymbol{r}_1$ in the three-body term of Eq. (7), we find the effective two-body potential

$$V_2^{\mathrm{eff}}(r) = -\frac{g^2}{2\pi}\ln r + V_2(r), \tag{8}$$

where we introduced the Coulomb coupling constant $g = 4\pi\alpha\sqrt{n/m}$ in the same way as in [1]. Now the potential energy can be rewritten in terms of the density variations $\delta\rho = \rho - n$, with the replacements $\rho(\boldsymbol{r}) \to \delta\rho(\boldsymbol{r})$ and $V_2(\boldsymbol{r}) \to V_2^{\mathrm{eff}}(\boldsymbol{r})$ made in Eq. (7). On large space scales the main term in the potential energy is the one containing $\ln r$, which leads to the existence of a plasmon gap in the longitudinal branch of the excitation spectrum: $\omega(q=0) = \omega_0 = \sqrt{ng^2/m}$.

Note that the ground-state wavefunction (5) is of the same form (up to the replacement $\alpha \to m/2$) as the absolute value of the Laughlin wavefunction for the fractional quantum Hall state, which is known [14] to produce a quasi-long-range order in the 1-particle density matrix written in an appropriately chosen gauge:

$$\rho(\boldsymbol{r},\boldsymbol{r}') \sim |\boldsymbol{r}-\boldsymbol{r}'|^{-m/2}. \tag{9}$$

This result was derived in [14] for Laughlin's liquid ground state which is expected to exist at not very large m. Within our model the ground state is of the form (5) for any α, including those corresponding to the quantum hexatic state. It might be interesting to note that the Girvin–MacDonald analysis can be repeated without any modifications for arbitrary values of α, so the result (9) is generally valid for our model.

Below we will be interested in the presence of superfluidity in the ground state defined in Eq. (5). We define the superfluid density n_s via the variation of the ground-state energy $\mathcal{E}_0$ (the reader should not be confused by the different quantities $\mathcal{E}_0$ and E^0 in Eq. (4)) with respect to the external (source) gauge field $\boldsymbol{a}$ minimally coupled to the Hamiltonian (4) via the replacement $\nabla_j \to \nabla_j + i\boldsymbol{a}(\boldsymbol{r}_j)$ in the kinetic energy term only (see, for instance, Ref. [15]):

$$\frac{n_s}{m} = \left.\frac{\partial^2 \mathcal{E}_0\{\boldsymbol{a}\}}{\partial \boldsymbol{a}_\alpha(\boldsymbol{q})\partial \boldsymbol{a}_\beta(-\boldsymbol{q})}\right|_{q\to 0} \cdot (\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta) = \frac{n}{m} - \mathcal{P}(q\to 0, \omega = 0), \tag{10}$$

where the polarization function $\mathcal{P}(q,\omega=0)$ is related to the transverse current–current correlation function $\Pi_\perp(q,\omega)$ by the zero-T fluctuation–dissipation theorem relation:

$$\mathcal{P}(q,\omega=0) = \frac{2}{\pi}\int_0^\infty \frac{\Pi_\perp(q,\omega)}{\omega} d\omega. \tag{11}$$

In Section 4 we will find the relation between the *instantaneous* transverse current–current correlation function $\Pi_\perp(q) = \frac{1}{\pi}\int_0^\infty \Pi_\perp(q,\omega)d\omega$ and the density–density correlation function. Note that the replacement $\nabla \to \nabla + i\boldsymbol{a}$ made in all terms in Eq. (4) would retain the SUSY structure of the theory and therefore would not change the ground-state energy $E_0 = 0$.

The ground-state wavefunction is given by Eq. (5) for any value of the coupling constant α, however this multiparticle wavefunction describes at least two ground states which are very different physically. Namely, for $\alpha \leqslant 1$ the ground state is a superfluid [8] and its superfluid density n_s is equal to the total particle density n. However, superfluidity is not guaranteed by the form of the ground-state wavefunction only: at very large values of α (larger than $\alpha_{\rm cr} \approx 35$) the ground state is expected to possess translational symmetry breaking realized in the formation of a triangular lattice [24] (in fact, only quasi-long Berezinsky-type [16] translational order is formed). Such a state does not possess superfluidity, $n_s = 0$. Therefore, at least one quantum phase transition, between the states with non-zero and zero n_s, should happen as the parameter α increases. There is also possible room for the second phase transition. To see it, note that the Galilei invariance of the Hamiltonian (4) implies (at $T = 0$) the absence of a normal density: $n_n = n - n_s = 0$. It means that non-zero n_n can appear due to spontaneous breakdown of Galilei invariance, which may happen simultaneously with the lattice formation (so that n_s jumps from n to zero), or at some other critical value $\alpha_G < \alpha_{\rm cr}$. Moreover, there are no special reasons for the superfluid density to

go to zero at $\alpha = \alpha_{cr}$, as it may happen at $\alpha \in [\alpha_G, \alpha_{cr}]$. Finally, there is one more symmetry, the orientational one, which can be spontaneously broken by the ground state at $\alpha = \alpha_O \in [\alpha_G, \alpha_{cr}]$. Below we start our consideration from the small-α range where no symmetry is broken, then proceed (in Section 7) to the region of largest $\alpha > \alpha_{cr}$, and at the end we present some speculations on the system's behavior in the intermediate range of α.

3. The special case $\alpha = 1/2$: Mapping to quantum mechanics of normal matrices

We will show now that the dynamics of a Gaussian ensemble of $N \times N$ complex matrices ($N \to \infty$) constrained by the condition $[M, M^+] = 0$ (which is the definition of a normal matrix) is intimately related to the many-body quantum mechanics defined by the Hamiltonian (4) with $\alpha = 1/2$. This relation is of the same origin as the by now well-known relation [9] between dynamics of the Dyson matrix ensembles and the Calogero–Sutherland model of many-body quantum mechanics in one dimension, where the coordinates of 1D particles are just eigenvalues of the corresponding matrices. Eigenvalues of a normal matrix are, in general, complex numbers, which can be considered as coordinates of particles living in a 2D space. Thus our goal here is to show how it is possible to perform the functional integral over most part of the variables characterizing the dynamics of a normal matrix and reduce the problem to the quantum mechanics for a set of complex eigenvalues $z_i, \bar{z}_i$. Our derivation closely follows the one presented in [9] (see also Ref. [17]) for three Dyson ensembles. Consider the partition function

$$Z = \int \mathcal{D}\hat{M}(\tau)\mathcal{D}\hat{M}^+(\tau) \exp[-S(\hat{M}, \hat{M}^+)] \prod_\tau \delta([\hat{M}(\tau), \hat{M}^+(\tau)]) \tag{12}$$

with the flat measure of integration and the Gaussian action

$$S(\hat{M}, \hat{M}^+) = \int \left[\frac{1}{4} \operatorname{Tr}\left(\frac{\partial \hat{M}^+}{\partial \tau}\frac{\partial \hat{M}}{\partial \tau}\right) + \operatorname{Tr}(\hat{M}^+\hat{M})\right] d\tau. \tag{13}$$

Note now that any normal matrix can be diagonalized by an appropriate unitary transformation: $\hat{M} = \hat{U}^+\hat{Z}\hat{U}$, where $\hat{Z}$ is diagonal with the elements z_i. Therefore we can change variables in the functional integral (12):

$$\mathcal{D}\hat{M}(\tau)\mathcal{D}\hat{M}^+(\tau) = J\{z_i\} \prod_i dz_i\, d\bar{z}_i\, \mathcal{D}\hat{U}, \tag{14}$$

where the Jacobian of the transformation is given [18] by the square of the absolute value of the Vandermonde determinant: $J\{z_i\} = \text{const}\,|\Delta\{z_i\}|^2 = \text{const}\,\prod_{i>j}|z_i - z_j|^2$. Consider now the Hamiltonian corresponding to the above functional integral in the transformed variables,

$$H = -\frac{2}{\sqrt{g}} \sum_i \left[\frac{\partial}{\partial z_i}\sqrt{g}\frac{\partial}{\partial \bar{z}_i} + \frac{\partial}{\partial \bar{z}_i}\sqrt{g}\frac{\partial}{\partial z_i}\right] + \sum_i |z_i|^2 + \tilde{H}. \tag{15}$$

Here the first sum over i constitutes a Laplacian operator in curved space with the square root of the determinant of the metric tensor $\sqrt{g} = |\Delta\{z_i\}|^2$, the second one comes from the "potential energy" $\mathrm{Tr}\,(\hat{M}^+\hat{M})$, and the last term corresponds to rotations of the angular (i.e. related to unitary transformations) variables, and thus vanishes when acting on any singlet ($\hat{U}$-invariant) state. The states with non-zero angular momenta have much higher energy [19] and thus are irrelevant, therefore we will omit $\tilde{H}$ below. Let us note now that the rest of the Hamiltonian (15) can be rewritten in the form

$$-\frac{4}{|\Delta\{z_i\}|}\sum_i \frac{\partial^2}{\partial z_i\,\partial \bar{z}_i}|\Delta\{z_i\}| + V\{z_i, \bar{z}_i\}, \tag{16}$$

where the potential energy V is given by

$$V\{z_i, \bar{z}_i\} = \sum_i \sum_{j\neq i, k\neq i} \frac{1}{(z_i - z_k)(\bar{z}_i - \bar{z}_j)} + 4\sum_{i\neq j} \frac{\partial^2}{\partial z_i\,\partial \bar{z}_i} \log|z_i - z_j| + \sum_i |z_i|^2. \tag{17}$$

After redefinition of the wavefunction $\Psi \to |\Delta\{z_i\}|\Psi$ and rewriting z_i and $\bar{z}_i$ via real 2-vectors $\boldsymbol{r}_i$, we are back to the Hamiltonian defined in Eq. (4) (apart from the constant term E^0) with $\alpha = 1/2$, $m = 1/2$ and $n = 1/\pi$. Evidently, the values of m and n just give the units of time and length in the problem, and can be rescaled arbitrary. The equilibrium probability distribution for the eigenvalues (for the density of eigenvalues equal to n) is

$$P\{z_i\} = |\Psi_0\{z_i\}|^2 = \mathrm{const}\cdot \prod_{j>k} |z_j - z_k|^2 \exp\left(-\pi n \sum_i r_i^2\right) \tag{18}$$

and coincides (up to a constant normalizing factor) with the corresponding distribution for the general complex ensemble [20,18] (cf. also Refs. [21,22]). Unfortunately it does not seem possible to establish an exact relation between the dynamics of general complex matrices and some Hamiltonian problem defined in terms of eigenvalues only; the problem is that a general complex matrix cannot be diagonalized by a unitary transformation and thus angular variables cannot be eliminated from the dynamical problem.[3] It is only the stationary distribution of its eigenvalues $P_{\mathrm{complex}}\{z_i\}$ which coincides (up to a constant factor) with such a distribution for a normal matrix ensemble given by Eq. (18). However, the relations obtained above indicate that the correspondence between the KKLZ Hamiltonian and normal matrices is much deeper than just on the level of the stationary distribution. The supersymmetric structure of this Hamiltonian and the obtained relation with a relatively simple matrix model lead us to believe that the model under discussion might be a rather direct generalization of the Calogero–Sutherland model for a 2D case, and thus might even be amenable for the exact solution. However,

[3] If this would be the case, one could try to solve the dynamical problem by the generalization of the method employed in [9], i.e. by the mapping onto the appropriate supersymmetric sigma model.

presently we are not aware of any exact way to treat normal matrix dynamics, and therefore will turn in the next sections to some more elementary methods, which still make it possible to find some exact relations between the equal-time correlation functions of the KKLZ model at arbitrary value of the parameter α.

4. The relation between the current–current and density–density correlation functions

The SUSY structure of the Hamiltonian (4) makes it possible to find, using the method used previously by Taniguchi, Shastry and Altshuler for the Calogero–Sutherland model [10] (see also Ref. [23]), an exact relation between the regular parts of the equal-time current–current $\Pi^{\mathrm{reg}}_{\alpha\beta}(\boldsymbol{r})$ and density–density $K(r)$ correlation functions defined as

$$\Pi^{\mathrm{reg}}_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}') = \langle J_\alpha(\boldsymbol{r}) J_\beta(\boldsymbol{r}')\rangle^{\mathrm{reg}}, \qquad K(\boldsymbol{r}-\boldsymbol{r}') = \langle \rho(\boldsymbol{r})\rho(\boldsymbol{r}')\rangle^{\mathrm{reg}}. \tag{19}$$

In the definitions (19) the singular (i.e. proportional to δ-functions) terms in the current and density correlators are excluded (i.e. $\boldsymbol{r} \neq \boldsymbol{r}'$). It means, in particular, that the correlator $K(\boldsymbol{r})$ is related to the inverse Fourier transform $s(\boldsymbol{r})$ of the static structure factor $S(\boldsymbol{p}) = \int (d\omega) S(\omega, \boldsymbol{p})$ as follows:

$$K(\boldsymbol{r}-\boldsymbol{r}') = \langle 0| \sum_{j\neq k} \delta(\boldsymbol{r}-\boldsymbol{r}_j)\delta(\boldsymbol{r}'-\boldsymbol{r}_k)|0\rangle = s(\boldsymbol{r}-\boldsymbol{r}') - n\delta(\boldsymbol{r}-\boldsymbol{r}') + n^2, \tag{20}$$

where n is the mean density of particles; the last term in (20) is due to the fact that $K(\boldsymbol{r})$ (contrary to $s(\boldsymbol{r})$) is the reducible correlation function). Similarly, the relation between the full current–current correlation function and its regular part is

$$\begin{aligned}\Pi^{\mathrm{reg}}_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}') = \frac{(-i)^2}{4m^2}\langle 0| \sum_{j\neq k} \left(\frac{\partial}{\partial r_{j,\alpha}}\delta(\boldsymbol{r}-\boldsymbol{r}_j) + \delta(\boldsymbol{r}-\boldsymbol{r}_j)\frac{\partial}{\partial r_{j,\alpha}} \right) \\ \times \left(\frac{\partial}{\partial r_{k,\beta}}\delta(\boldsymbol{r}'-\boldsymbol{r}_k) + \delta(\boldsymbol{r}'-\boldsymbol{r}_k)\frac{\partial}{\partial r_{k,\beta}} \right) |0\rangle \\ = \Pi_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}') - \frac{n}{m} E_{\mathrm{kin}} \delta_{\alpha\beta}\delta(\boldsymbol{r}-\boldsymbol{r}'),\end{aligned} \tag{21}$$

where $E_{\mathrm{kin}} = \frac{m}{2N}\sum_i \langle 0|v_i^2|0\rangle$ is the mean ground-state kinetic energy per particle (in D dimensions the last term in Eq. (21) should be multiplied by $2/D$).

The trick to prove the relation between $\Pi^{\mathrm{reg}}_{\alpha\beta}(\boldsymbol{r})$ and $K(\boldsymbol{r})$ is to substitute, following Ref. [10], the standard definition of the current operator by the equivalent expression

$$J_\alpha(\boldsymbol{r}) = \frac{1}{2m}\sum_i \left(q^+_{i,\alpha}\delta(\boldsymbol{r}-\boldsymbol{r}_i) + \delta(\boldsymbol{r}-\boldsymbol{r}_i)q_{i,\alpha} \right). \tag{22}$$

Then, using the commutation relations

$$[q_{i,\alpha}, q^{+}_{j,\beta}] = 4\alpha \left[(1-\delta_{ij}) D_{\alpha\beta}(\boldsymbol{r}_i - \boldsymbol{r}_j) - \delta_{ij} \sum_{l \neq i} \left(D_{\alpha\beta}(\boldsymbol{r}_i - \boldsymbol{r}_l) - \pi n \delta_{\alpha\beta} \right) \right] \quad (23)$$

with $D_{\alpha\beta}(\boldsymbol{r}) = \nabla_\alpha \nabla_\beta \ln r$, and identities $q_{i,\alpha}|0\rangle = 0$, we get

$$\Pi^{\mathrm{reg}}_{\alpha\beta}(\boldsymbol{r}) = \frac{\alpha}{m^2} \frac{\delta_{\alpha\beta} - 2\hat{r}_\alpha \hat{r}_\beta}{r^2} K(\boldsymbol{r}). \quad (24)$$

The relation (24) is applicable for any value of the coupling constant α (remember that $\alpha \ll 1$ corresponds to the superfluid ground state, whereas at very large $\alpha \geqslant \alpha_{\mathrm{cr}} \approx 35$ the Wigner-type crystal is formed [24]). One may wonder whether it is allowed to use the relation (24) at $\alpha \geqslant \alpha_{\mathrm{cr}}$ since in the course of its derivation we have used space homogeneity of the correlation functions, which usually is not present in the crystalline state; however we will see below that the "crystalline" state present in our problem actually does not possess a true positional long-range order and thus does not break space homogeneity.

5. Transverse currents and "instantaneous" magnetic response

We are going to obtain the momentum-space representations for the longitudinal and transverse current–current correlators defined as $\Pi_{\alpha\beta}(\boldsymbol{q}) = \hat{q}_\alpha \hat{q}_\beta \Pi_{\parallel}(\boldsymbol{q}) + (\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta)\Pi_{\perp}(\boldsymbol{q})$. Performing the Fourier transformation of the expression (24) and using the relation (21), we get

$$\Pi_{\alpha\beta}(\boldsymbol{q}) = \frac{2\pi\alpha}{m^2} \int \frac{d^2p}{(2\pi)^2} \frac{(p+q)_\alpha (p+q)_\beta}{(\boldsymbol{p}+\boldsymbol{q})^2} \tilde{K}(\boldsymbol{p}) + \frac{2\pi\alpha}{m^2} n^2 \frac{q_\alpha q_\beta}{q^2} + \frac{n}{m} E_{\mathrm{kin}} \delta_{\alpha\beta}. \quad (25)$$

Here $\tilde{K}(\boldsymbol{p}) = S(\boldsymbol{p}) - n$ is the Fourier transform of $K(\boldsymbol{r}) - n^2$. In order to obtain the transverse polarization function $\Pi_{\perp}(\boldsymbol{q})$ one has to contract Eq. (25) with the transverse projector $\delta_{\alpha\beta} - \hat{q}_\alpha \hat{q}_\beta$:

$$\Pi_{\perp}(\boldsymbol{q}) = \frac{2\pi\alpha}{m^2} \int \frac{d^2p}{(2\pi)^2} \left[1 - \frac{[(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{q}]^2}{(\boldsymbol{p}+\boldsymbol{q})^2 q^2} \right] \tilde{K}(\boldsymbol{p}) + \frac{n}{m} E_{\mathrm{kin}}. \quad (26)$$

The $q \to 0$ limit of the frequency-dependent $\Pi_{\perp}(\boldsymbol{q}, \omega)$ is zero for any ω, leading to the same result for the equal-time correlator: $\Pi_{\perp}(\boldsymbol{q} \to 0) = 0$. This condition makes it possible to determine E_{kin}:

$$E_{\mathrm{kin}} = -\frac{\pi\alpha}{mn} \int \frac{d^2p}{(2\pi)^2} \tilde{K}(\boldsymbol{p}) = -\frac{\pi\alpha}{mn} \left[K(r=0) - n^2 \right]. \quad (27)$$

Now, $K(r=0) = 0$ since the ground state (5) forbids two particles to be at the same point: $\Psi_0(\ldots, \boldsymbol{r}, \ldots, \boldsymbol{r}, \ldots) = 0$. Consequently, the expectation value of the kinetic energy in the ground state is given by

$$E_{\rm kin} = \frac{\pi\alpha n}{m} = \frac{\omega_0}{4}, \tag{28}$$

where $\omega_0 = \sqrt{ng^2/m}$ is the plasma frequency. The relation (28) was checked numerically (see Appendix A for details) and found to be obeyed with good accuracy for all values of α, including those where quantum hexatic phase is formed and both the Galilei and the rotational symmetries are broken. We will discuss the implications of this result below, in Section 7.

Then, Eq. (26) may be rewritten as

$$\Pi_\perp(\boldsymbol{q}) = -\frac{2\pi\alpha}{m^2}\int \frac{d^2p}{(2\pi)^2}\left[\frac{[(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{q}]^2}{(\boldsymbol{p}+\boldsymbol{q})^2q^2} - \frac{1}{2}\right](S(\boldsymbol{p}) - n)\,. \tag{29}$$

If the orientational symmetry is unbroken (which is the case for not very large α), the correlation function $S(\boldsymbol{p})$ depends on the absolute value of the vector $\boldsymbol{p}$ only. Then the angular integral in Eq. (29) can be easily calculated, the result is

$$\Pi_\perp(q) = -\frac{\alpha}{2m^2}\int_0^q (S(p) - n)\left(1 - \frac{p^2}{q^2}\right) p\,dp. \tag{30}$$

At small momenta p the structure factor $S(p)$ goes to zero: $S(p \to 0) = np^2/2m\omega_0$, therefore the low-momenta limit of the transverse correlator reduces to

$$\Pi_\perp(q \to 0) = \frac{\alpha n}{8m^2}\cdot q^2. \tag{31}$$

In the high-q limit the integral in Eq. (30) reduces to the one of Eq. (27):

$$\Pi_\perp(q \to \infty) = \frac{\pi\alpha n^2}{m^2}. \tag{32}$$

In the special case $\alpha = 1/2$ the current–current correlation function can be calculated explicitly. Indeed, the real-space density–density correlation function is known [20,18],

$$K_{1/2}(r) = n^2[1 - \exp(-\pi n r^2)] \tag{33}$$

(this expression was derived by direct integration of the total distribution function (18) over the positions of all but two particles). The corresponding structure factor is

$$S_{1/2}(p) = n[1 - \exp(-p^2/4\pi n)], \tag{34}$$

which leads, via integration in Eq. (30), to

$$\Pi_{\perp,1/2} = \frac{\pi n^2}{2m^2}\left[1 - \frac{1 - \exp(-q^2/4\pi n)}{q^2/4\pi n}\right]. \tag{35}$$

Note that the current–current correlations in the Calogero–Sutherland model were shown to be related to the correlations of eigenfunctions of the corresponding random matrices [10]; we expect that a similar relation might exist between the correlation function (35) and some eigenfunctions correlations in the random ensemble of normal matrices.

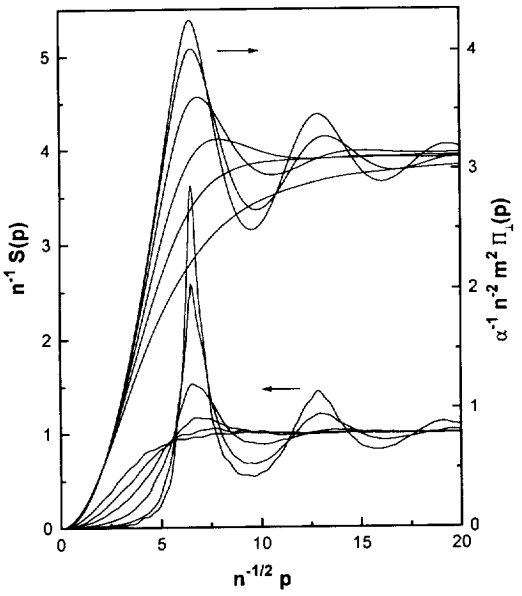

Fig. 1. The structure factor $S(p)$ (left axis) and the transverse current–current correlator $\Pi_\perp(p)$ (right axis) computed numerically by the mapping of the Schrödinger equation with the Hamiltonian (4) onto a Langevin dynamics of classical particles (see Appendix A). Shown are the curves for $\alpha = 0.5$, 1, 2, 5, 15 and 30. The increase of α corresponds to the increase of the peak height at $p \sim 2\pi n^{1/2}$.

We have employed numerical simulations (cf. Appendix A) to obtain the structure factor $S(p)$ for several different values of α, and then used the obtained results to compute, using Eq. (30), the transverse current–current correlation function $\Pi_\perp(p)$. The results are shown in Fig. 1. The peaks at $p \sim 2\pi n^{1/2}$ correspond to a roton minimum in the excitation spectrum that becomes deeper as α increases.

6. Longitudinal currents and low-q plasmon dispersion

Now we turn to the calculation of the longitudinal correlator $\Pi_\parallel(\boldsymbol{q})$. Contracting Eq. (25) with the longitudinal projector and using (27), we obtain

$$\Pi_\parallel(\boldsymbol{q}) = \frac{2\pi\alpha}{m^2}n^2 - \Pi_\perp(\boldsymbol{q}) = \frac{2\pi\alpha}{m^2}n^2 + \frac{\alpha}{2m^2}\int_0^q (S(p) - n)\left(1 - \frac{p^2}{q^2}\right) p\, dp. \quad (36)$$

Alternatively, the value $\Pi_\parallel(0)$ may be found with the help of the continuity equation that leads to the relation $q^2\Pi_\parallel(q,\omega) = \omega^2 S(q,\omega)$ for the frequency-dependent correlators. At $q \to 0$ the whole spectral weight is saturated by the plasmon excitations with $\omega = \omega_0$, so we get

$$q^2\Pi_\parallel(q) = \int \frac{d\omega}{2\pi}\omega^2 S(q,\omega) \xrightarrow{q\to 0} \frac{nq^2\omega_0}{2m} \quad (37)$$

leading to $\Pi_{\parallel}(0) = n\omega_0/2m$, in agreement with (36).

Eq. (36) can be used to find the low-q dispersion of the plasmon energy $\omega(q)$: using the continuity equation $q^2\Pi_{\parallel}(q,\omega) = \omega^2 S(q,\omega)$ together with the f-sum rule $\int (d\omega)\omega S(q,\omega) = nq^2/2m$, we convert Eq. (36) into the form

$$\frac{\partial}{\partial q^2}\tilde{\omega}(q) = \frac{\alpha}{mn}\frac{1}{q^4}\int_0^q (S(p) - n)p^3\, dp, \tag{38}$$

where

$$\tilde{\omega}(q) = \frac{\int \omega^2 S(q,\omega)d\omega}{\int \omega S(q,\omega)d\omega}. \tag{39}$$

As long as the spectrum $\omega(q)$ is non-decayable (i.e. $\mathrm{Im}\,\omega(q) = 0$), $\tilde{\omega}(q)$ coincides with $\omega(q)$.

Finally, using the relation $S(q) = nq^2/2m\omega(q)$ valid for $q \to 0$ and neglecting small $S(q)$ compared to n in the r.h.s. of Eq. (38), we arrive at

$$\tilde{\omega}(q) - \omega_0 = -\alpha\frac{q^2}{4m}. \tag{40}$$

Eq. (40) is very similar to the one derived [26] in the lowest-order perturbation theory (over α) for the purely Coulomb 2D Bose liquid (the result of Ref. [26] contains an extra numerical factor ≈ 0.8 in the r.h.s. of Eq. (40)). Let us emphasize, however, that in the present model we are able to derive the relation (40) for *any* value of α, the only limitation being the low-q limit. In the field-theoretic language, the low-momentum excitation spectrum of the KKLZ Hamiltonian is given by the first-loop term in the diagram expansion. We believe that this remarkable property is due to the supersymmetry of the model, leading somehow to the cancellation of all higher-order diagrams. It is not yet clear to us whether the one-loop calculation is exact not only for the low-q spectrum, but for the spectrum and the correlation functions in general. To study this issue one needs to develop an explicitly supersymmetric second-quantized representation and the corresponding diagram technique for the KKLZ model.

As long as $\alpha \leqslant 1$, the quadratic dispersion $\omega(q)$ given by Eq. (40) produces a relatively weak minimum between ω_0 and the free-particle limit $\omega_{\mathrm{free}}(q) = q^2/2m$. Such a spectrum is consistent with a totally superfluid ($n_s = n$) ground state. With the increase of α the minimum in the spectrum becomes deeper; for a qualitative description of the spectrum one can use a kind of interpolation formula as

$$\omega^2(q) \approx \omega_0^2 - \omega_0\alpha q^2/2m + (q^2/2m)^2. \tag{41}$$

The minimum of the r.h.s. of Eq. (41) touches zero at $\alpha = 2$. It can be considered as an indication of a possible spectrum instability and phase transition(s) in the large-α region. However, the nature of Eq. (41) as interpolation formula of unknown (at large α) accuracy makes it impossible to use it for any exact conclusion about the behavior of the spectrum. In the next section we will try to consider this problem starting from the opposite limit of extremely large α.

7. "Crystalline" ground state: Quantum hexatic without shear modulus

For very large values of $\alpha > \alpha_{cr} \approx 35$ the ground state is expected to possess symmetry breaking realized in the formation of triangular lattice [24]. The static structure factor $S(p)$ can be determined knowing the ground-state wavefunction only. Its calculation involves integration over all the particle's coordinates $\boldsymbol{r}_i$ with the weight given by $\Psi_0^2\{\boldsymbol{r}_i\}$, which is formally equivalent to the calculation of the structure factor of a classical 2D crystal with a logarithmic repulsion between the particles, $V(r) = 4\alpha \cdot \ln r$, and "temperature" $\mathcal{T} = 1$ (cf. Appendix A). Such a classical crystal has a finite shear modulus $\mu_{cl} = \alpha n/2$ (cf. Ref. [25]; here we present the result in our notations) and is expected to undergo dislocation-mediated melting when α decreases below some critical value α_1. For $\alpha > \alpha_1$ it is stable with respect to dislocation formation, but does not possess long-range translational order (as any other 2D crystal at finite temperature). It means that the structure factor $S(p)$ does not have Bragg peaks near the principal inverse lattice vectors $\boldsymbol{p}_a$, but shows only power-law singularities,

$$S(p) \sim |\boldsymbol{p} - \boldsymbol{p}_a|^{\eta - 2} \tag{42}$$

with $\eta = \alpha^{-1} \cdot 4\pi/\sqrt{3}$. Let us now go back to our original problem, which is a quantum 2D crystal. Suppose that it has finite shear modulus μ (now without subscript 'cl'), then the spectrum of the transverse sound would be linear, as usual: $\omega(q) = q \cdot \sqrt{\mu/mn}$, where q is the wavevector of a shear deformation on the top of the crystal state. (In terms of the momenta $\boldsymbol{p}$ of the original Bose particles it corresponds to $\omega_\perp(\boldsymbol{p}) \sim |\boldsymbol{p} - \boldsymbol{p}_a|$ for any a). Now we can calculate the zero-point fluctuations of lattice displacements $\langle u^2 \rangle \sim \int d^2q\, \omega^{-1}(q)$ and find that it is finite, which means a finite Debye–Waller factor and thus the presence of Bragg peaks in the structure factor $S(p)$ – in contradiction with the above-obtained equation (42). The only way out of this contradiction is to suppose that the quantum solid with our very special interaction given by Eq. (4) has actually *zero* shear modulus: $\mu = 0$. More precisely, we need to suppose that $\omega(q) \propto q^2$, which leads then to a logarithmically divergent $\langle u^2 \rangle$, in agreement with Eq. (42). It means that the "crystalline" ground state expected to exist for $\alpha > \alpha_1$ should be more adequately called "quantum hexatic" (QHX), which is (in close analogy to the classical hexatic phase introduced by Halperin and Nelson within their picture of dislocation-mediated thermal melting of 2D crystals [11]) the phase without shear modulus but still with anisotropy of the excitation spectrum:

$$\omega_\perp(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{p}_a)^2/2M \tag{43}$$

(where M is some parameter with the dimension of mass), whereas in the isotropic liquid excitation spectrum depends on the momentum absolute value $|\boldsymbol{p}|$ only.

The absence of a shear modulus in the high-α ground state of the Hamiltonian (4) can also be shown by direct calculation, which can be conveniently performed representing the potential energy of the interaction as

$$U_{\mathrm{KKLZ}} = \frac{2\alpha^2}{m}\sum_i\left(\sum_{j\neq i}\frac{\boldsymbol{r}_j-\boldsymbol{r}_i}{(\boldsymbol{r}_j-\boldsymbol{r}_i)^2}\right)^2 + \frac{2\pi\alpha}{m}\sum_{i\neq j}\delta(\boldsymbol{r}_i-\boldsymbol{r}_j) + \frac{(2\pi\alpha n)^2}{2m}\sum_i r_i^2. \tag{44}$$

Now one can calculate the variation of the potential energy (actually the last two sums in (44) do not contribute) corresponding to an infinitesimal uniform shear deformation $\epsilon_{\alpha\beta}$ of the regular lattice. Then one finds that it does not contain the term proportional to $\epsilon_{\alpha\beta}^2$, i.e. the expansion starts from the term $(\nabla\epsilon_{\alpha\beta})^2$, which means that $\mu(q) \propto q^2$, and, therefore, $\omega(q) \propto q^2$. These calculations are presented in Appendix B, where we also find the effective mass M entering Eq. (43):

$$M = \frac{2m}{\alpha}. \tag{45}$$

As could be expected, the excitation spectrum (43) becomes harder (i.e. the effective mass M decreases) with increasing α.

The absence of a genuine crystalline state in the KKLZ model at arbitrary strong interaction $\alpha \to \infty$ stems from its very special form and certainly is not present in a generic 2D Bose system. However, in a more realistic system QHX may exist as an intermediate ground state realized at moderately large repulsion.

Let us consider now the response of the QHX state to a magnetic field h. Since particles in this state are almost localized, we can neglect (in a first approximation) the multi-particle exchange contributions and estimate the susceptibility as $\chi \sim \frac{n}{m}\langle u^2\rangle \sim \frac{M}{m}\ln(nL^2)$ (where L is the system's size). Comparing this expression with the magnetic susceptibility of a superfluid, which is proportional to $n_s L^2$, one concludes that $n_s = 0$ in the QHX state.

Now we need to address some apparent paradox: (i) according to Eq. (11), the normal density $n_n = n - n_s$ is proportional to the integral $\int_0^\infty \Pi_\perp(q \to 0, \omega)d\omega/\omega$ where the expression under the integral is positive; (ii) the equal-time current–current correlation function $\Pi_\perp(q) = \int_0^\infty \Pi_\perp(q,\omega)d\omega/\pi$ vanishes at $q \to 0$ (cf. Eq. (28)); (iii) then, how is it possible to get a non-zero n_n (and even cancel n_s completely)? In fact, to get a non-zero n_n we need to allow for the singular behavior of the function $\Pi_\perp(q,\omega)$ at low q, ω (in other words, the limiting procedure $q \to 0$ and integration in Eq. (11) should not commute with each other). As a simple example one can consider $\Pi(q,\omega) \propto f(q^2/m\omega)$ with $f(x) = x/(1+x^2)$. However, such a behavior of $\Pi(q,\omega)$ is only possible if there are some gapless excitations in the system, which is just the case in the QHX state (cf. Eq. (43)). In general, the appearance of gapless excitations with non-zero momentum (low-momenta excitations in our system always have a large plasmon gap ω_0) may only be a result of a phase transition, which is associated with the spontaneous breakdown of Galilei invariance. Generally speaking, Galilei invariance may be spontaneously broken at $\alpha = \alpha_G < \alpha_{\mathrm{cr}}$, which would mean the existence of two phase transitions and some intermediate state between the SF and the QHX states.

8. Between superfluidity and quantum hexatic: Speculations

We found in previous sections that at small α the ground state Ψ_0 possesses total superfluidity ($n_s = n$) and the excitation spectrum beyond the ground state is gapful (cf. Eq. (41)). At very large $\alpha \geqslant \alpha_{cr} \approx 35$ the ground state is not superfluid at all, $n_s = 0$, and the excitation spectrum is very soft (cf. Eq. (43)). How do(es) the transition(s) between these two qualitatively different ground states occur? The most trivial possibility would be that of a direct first-order phase transition at $\alpha = \alpha_{cr}$. This would also mean that the first-order melting/crystallization transition happens in the classical 2D Coulomb gas at the interaction parameter $\Gamma = 4\alpha \approx 140$. However, the persistence of total superfluidity up to $\alpha \approx 35$ looks rather improbable since at so large values of α the characteristic energy of the interaction between particles, $g^2/2\pi$, exceeds the kinetic energy $E_{kin} = \pi\alpha n/m$ by more than 2 orders of magnitude. Note also that the above-cited value of E_{kin} is about 100 times larger than the naive estimate n/m; it means that particles are almost localized in rather small regions of space (much smaller than the mean interparticle distance $n^{-1/2}$) and diffuse from these regions on much longer time scales only. Therefore we suspect that at least two phase transitions take place between the totally SF state and the QHX state.

As we discussed above, the QHX quantum ground state is seen as a 2D Berezinsky crystal with logarithmic interaction as long as only the equal-time density correlations are considered. The nature of melting in such a 2D classical crystal is not quite clear. Old numerical simulations reported a weak first-order transition [24], but recent simulations on 2D systems with other types of interactions [27] reveal that to discern between first- and second-order melting one probably would need to simulate systems with a huge number of particles, which was not yet done for the system of interest here.

If melting of such a 2D crystal occurs via the Halperin–Nelson mechanism, then there exists a usual hexatic state which possesses quasi-long-range orientational order, i.e. power-law decay of the orientational order parameter,

$$\langle \exp[6i(\theta(0) - \theta(\boldsymbol{r}))] \rangle \propto r^{-\eta_6}, \tag{46}$$

without any translational order. The corresponding quantum ground state would be characterized, apart from orientational quasi-LRO, by the isotropic structure factor $S(p)$ and by the existence of some gapless mode related with a slow rotation of the $\theta(\boldsymbol{r})$ angle. The other possibility is that the melting of the 2D crystal with logarithmic interaction occurs via a first-order phase transition, without hexatic phase; even in this case there is a room for a phase transition related to the breakdown of Galilei invariance, which would be manifested by the appearance of the gapless mode in the excitation spectrum. Indeed, the absence of the normal density n_n in the ground state of the system with purely potential interactions is the consequence of Galilei invariance of the Hamiltonian, which is certainly respected in the low-α limit. The appearance of non-zero n_n is thus related to spontaneous breakdown of Galilei invariance, which may occur at some value of α which we denote as α_G. As explained above, we expect $\alpha_G < \alpha_{cr}$, although we still cannot prove this inequality.

To conclude this section: if there exists any additional ground state sandwiched between SF and QHX, then the spectrum of excitations beyond such a state contains a gapless mode with isotropic spectrum: $\omega(p) \propto (p - p_0)^\beta$. Such a mode may be orthogonal to the density mode and, thus, it may be invisible in the usual structure factor $S(q, \omega)$.

9. Conclusions

We have studied the model Hamiltonian of the two-dimensional strongly interacting Bose liquid, which can be considered as a natural generalization of the Calogero–Sutherland Hamiltonian for the 2D case. This Hamiltonian is shown to possess non-relativistic supersymmetry. It enables us to derive a number of exact relations between the equal-time density–density and current–current correlation functions, and to find exactly the low-momentum excitation spectrum of the model, which is given by the one-loop correction to the mean-field result. The ground state is shown to exist in at least two qualitatively different phases as a function of the interaction parameter α: the superfluid state at small α and the quantum hexatic state at very large α. Arguments are presented in favor of the existence of a third state, intermediate between the two above-mentioned ones. For the special case of $\alpha = 1/2$ the considered model is shown to be equivalent to the quantum mechanics of normal complex matrices with a Gaussian weight, in exactly the same way as the Calogero–Sutherland model at the special values of its coupling constant is equivalent to the dynamics of the Wigner–Dyson matrix ensembles.

The model we have studied seems to be very interesting as a natural approximation for the problem of 2D Bose liquid with logarithmic pair-wise interaction, which is widely considered in relation with the composite-boson approach to the description of the fractional quantum Hall state [8,29], as well as in relation with statistics of vortex lines in HTSC superconductors [1,7]. The key physical problem here is the problem of the existence of a non-superfluid liquid ground state. We have shown within the considered model the existence of at least one non-trivial ground state, which is quantum hexatic, i.e. anisotropic non-superfluid liquid. It is still an open question if the destruction of superfluidity may occur without *any* explicit symmetry breaking in the real space, i.e. if it might be that the only spontaneously broken symmetry is Galilei invariance. To answer this question it is necessary to study the spectrum of the lowest-energy excitations at large wavevectors $q \sim 2\pi\sqrt{n}$ and at large $\alpha \in [2, 35]$ and look for the appearance of some gapless mode. Note that such a mode, if it exists, may be orthogonal to the density-oscillation mode, and thus may be absent from the dynamic structure factor $S(q, \omega)$.

The exact results obtained make it possible to suspect that the considered model can actually be exactly solvable, at least at some special values of α. There are several directions in which one could try to extend and develop the results presented here: (i) to find the time-dependent correlation functions, at least for $\alpha = 1/2$, by the solu-

tion of the normal-matrix problem; (ii) to generalize the correspondence between our Hamiltonian and the normal matrix problem for other values of α; (iii) to develop an explicitly supersymmetric second-quantized many-body theory and the corresponding diagram technique for our Hamiltonian; this might be very helpful in order to generalize our result for the spectrum, Eq. (40), for arbitrary momenta q. A completely different direction of study of this model is provided by its equivalence to a Fokker–Planck equation of multi-particle stochastic dynamics (cf. Appendix A), which makes it possible to use *classical* Monte Carlo simulations in order to study some properties of the quantum many-body problem in question.

Acknowledgements

We are grateful to V.B. Geshkenbein, A. Gorsky, A.I. Larkin, A.Yu. Morozov, A.M. Polyakov and A.S. Schwartz for useful discussions. We are especially indebted to L.B. Ioffe for his collaboration on earlier stages of this research and many useful advises. The hospitality of the Isaac Newton Institute where this study was initiated during the visit of one of us (M.V.F., August 1994) is gratefully acknowledged. This research was supported by the INTAS-RFBR grant # 95-0302, Swiss National Science Foundation collaboration grant # 7SUP J048531, and DGA grant # 94-1189 (M.V.F.).

Appendix A. SUSY, Langevin dynamics and numerical simulations

The relation between supersymmetric quantum mechanics and the Fokker–Planck equation of stochastic classical dynamics is well known now [13,28] (cf. Ref. [23]). In this appendix we will show how to use that equivalence in order to implement a fast numerical simulation of the many-body quantum-mechanical problem we are interested in.

Consider a system of N coupled Langevin equations:

$$\gamma \frac{dr_{j,\alpha}}{dt} = -\frac{\partial W\{\boldsymbol{r}_i\}}{\partial r_{j,\alpha}} + \xi_{j,\alpha}(t) \tag{A.1}$$

with a Gaussian distribution of random forces $\xi(t)$:

$$\overline{\xi_{j,\alpha}(t)\xi_{k,\beta}(t')} = 2\gamma \mathcal{T} \delta_{jk}\delta_{\alpha\beta}\delta(t-t'). \tag{A.2}$$

Here $\mathcal{T}$ is temperature of a thermal bath, γ is a damping coefficient which can be set to 1 by a proper rescaling of time.

The Fokker–Planck equation for the probability density function,

$$\gamma \frac{\partial \mathcal{P}\{\boldsymbol{r}_i\}}{\partial t} = \sum_{j,\alpha} \frac{\partial}{\partial r_{j,\alpha}} \left(\frac{\partial W}{\partial r_{j,\alpha}} + \mathcal{T} \frac{\partial}{\partial r_{j,\alpha}} \right) \mathcal{P}\{\boldsymbol{r}_i\}, \tag{A.3}$$

after the substitution $\mathcal{P}\{\boldsymbol{r}_i\} = e^{-W/2\mathcal{T}}\Psi\{\boldsymbol{r}_i\}$ reduces to an imaginary-time Schrödinger equation

$$\gamma\frac{\partial\Psi}{\partial t} = \sum_{j,\alpha}\left\{\mathcal{T}\frac{\partial^2\Psi}{\partial r_{j,\alpha}^2} + \left[\frac{1}{2}\frac{\partial^2 W}{\partial r_{j,\alpha}^2} - \frac{1}{4\mathcal{T}}\left(\frac{\partial W}{\partial r_{j,\alpha}}\right)^2\right]\Psi\right\}. \tag{A.4}$$

The many-particle ground state of Eq. (A.4) has zero energy, which is a consequence of a supersymmetry [13], and wavefunction $\Psi_0 = \text{const}\, e^{-W\{\boldsymbol{r}\}/2\mathcal{T}}$.

The equivalence of the Langevin, Fokker–Planck and Schrödinger approaches implies that the averaging over the many-particle ground state Ψ_0 is equivalent to averaging over the Langevin dynamics of particles (with respect to random force and initial particles distribution) [23]:

$$\langle\Psi_0|F\{\boldsymbol{r}_i\}|\Psi_0\rangle = \overline{\langle F\{\boldsymbol{r}_i\}\rangle}_L. \tag{A.5}$$

The relation (A.5) is very important as it enables us to compute ground state averages by numerical simulation of Langevin dynamics with a proper interparticle interaction W which is technically substantially simpler than direct quantum Monte Carlo simulations. The problem at hand, given by the Hamiltonian (4) and the wavefunction (5), corresponds to a stochastic classical dynamics (defined by Eq. (A.4)) of a one-component 2D Coulomb gas governed by the following potential energy:

$$W\{\boldsymbol{r}_i\} = -4\alpha\mathcal{T}\sum_{j>k}\ln|\boldsymbol{r}_j - \boldsymbol{r}_k| + 2\pi\alpha\mathcal{T}n\sum_i r_i^2. \tag{A.6}$$

where the parameters γ and $\mathcal{T}$ are related via $\gamma = 2m\mathcal{T}$. We have performed Monte Carlo computer simulations of the Brownian dynamics (A.1) with the potential (A.6) for the systems of $N = 109$ particles with different values of the parameter α. The results obtained for the structure factor $S(p)$ are shown in Fig. 1. Also shown are the curves for the correlation function of transverse currents, $\Pi_\perp(p)$, calculated from $S(p)$ according to Eq. (30).

We have also used this simulation in order to check Eq. (28) for different values α. Note that the general relation (A.5) can not be applied to compute the mean value of the kinetic energy since the operator of the kinetic energy contains space derivatives that have no immediate meaning in the classical Langevin dynamics. Fortunately, instead of E_{kin} one can compute the mean ground-state potential energy E_{pot} using the fact that the ground-state energy is zero. E_{pot} may be easily obtained by Langevin simulations using Eq. (A.5) since the operator of the potential energy $U_{\text{KKLZ}} + E^0$ contains particles coordinates only.

Numerical simulations have shown that Eq. (28) holds for all values of α from $\leqslant 1$, which corresponds to the SF state, to 100, corresponding to the CR state, within the precision of computation.

Appendix B. Elasticity theory for the quantum hexatic state

The most direct way to calculate the energy of the shear deformation would be to use Eq. (44) and to compare the energies of an ideal and the weakly deformed triangular lattices. However, this is rather inconvenient way in our case, since the lattice sums appearing in such a calculation are formally divergent due to the long-range nature of the interaction between the particles. Instead, we make use of the general relation between our quantum problem and the corresponding classical problem, as discussed in Appendix A.

Consider Eq. (A.4). It provides us with the general relation between the functionals of the "classical" and "quantum" potential energies $W[\boldsymbol{r}_j]$ and $U_{\mathrm{KKLZ}}[\boldsymbol{r}_j]$ (cf. Eqs. (A.6), (44); below we choose $\mathcal{T} = 1$, $\gamma = 2m$). In particular, we are allowed to consider a configuration of particles which corresponds to a slightly perturbed triangular lattice. Note that the term with a second derivative in $U_{\mathrm{KKLZ}}[\boldsymbol{r}_j]$ can be neglected, since it is a collection of δ-functions and thus does not contribute to the lattice sums. Now we may write the energy of elastic deformation as

$$U[\boldsymbol{u}(\boldsymbol{r})] = \frac{n}{4\gamma}\int d^2r(\delta W/\delta \boldsymbol{u})^2. \tag{B.1}$$

Instead of a direct calculation of $U[\boldsymbol{u}]$ for a given type of deformation, we use Eq. (B.1) together with the general expression [30] for the variation

$$n\frac{\delta W}{\delta u_\alpha} = \frac{\partial \sigma_{\alpha\beta}}{\partial r_\beta}, \tag{B.2}$$

where $\sigma_{\alpha\beta} = 2\mu_{\mathrm{cl}}u_{\alpha\beta} + \lambda_{\mathrm{cl}}u_{\gamma\gamma}\delta_{\alpha\beta}$ is the stress tensor, $u_{\alpha\beta}$ is the deformation tensor, and μ_{cl}, λ_{cl} are the shear and compression moduli calculated for the classical 2D crystal with the potential energy $W[\boldsymbol{r}_j]$; the presence of an extra (compared to the standard definition [30]) factor n in Eq. (B.2) is due to the fact that we define here $W[\boldsymbol{u}]$ as the total energy of the system. Now consider a purely shear deformation ($u_{\beta\beta} = 0$), and note that Eq. (B.1) reduces to

$$U[\boldsymbol{u}] = \frac{\mu_{\mathrm{cl}}^2}{\gamma n}\int d^2r(\partial u_{\alpha\beta}/\partial r_\beta)^2 = \frac{\mu_{\mathrm{cl}}^2}{4\gamma n}\int d^2r(\nabla^2 u_\alpha)^2. \tag{B.3}$$

Combining Eqs. (B.3) with the expression for the "classical" shear modulus $\mu_{\mathrm{cl}} = n\alpha/2$ from Ref. [25], we find the effective q-dependent shear modulus of the QHX state,

$$\mu(q) = \frac{\alpha^2 n}{16m}q^2. \tag{B.4}$$

Finally, the frequency of the shear mode can be obtained by generalizing the usual [30] relation $\omega(q) = q\sqrt{\mu/mn}$ for the q-dependent shear modulus, which leads to

$$\omega(q) = \alpha\frac{q^2}{4m}. \tag{B.5}$$

Comparing with Eq. (43) we find the value of the effective mass M given by Eq. (45).

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