ISSN 0021-3640, JETP Letters, 2020, Vol. 112, No. 6, pp. 376–382.
Published in Russian in Pis'ma v Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki, 2020, Vol. 112, No. 6, pp. 394–400.

NONLINEAR PHENOMENA

Lyapunov Exponent for Whitney's Problem with Random Drive

N. A. Stepanov[a, b, c, *] **and M. A. Skvortsov**[a, b, **]

[a] *Skolkovo Institute of Science and Technology, Moscow, 121205 Russia*

[b] *Landau Institute for Theoretical Physics, Russian Academy of Sciences, Chernogolovka, Moscow region, 142432 Russia*

[c] *National Research University Higher School of Economics, Moscow, 101000 Russia*

**e-mail: stepanov@itp.ac.ru*

***e-mail: skvor@itp.ac.ru*

Received August 20, 2020; revised August 20, 2020; accepted August 20, 2020

We consider the statistical properties of a non-falling trajectory in the Whitney problem of an inverted pendulum excited by an external force. In the case where the external force is white noise, we recently found the instantaneous distribution function of the pendulum angle and velocity over an infinite time interval using a transfer-matrix analysis of the supersymmetric field theory. Here, we generalize our approach to the case of finite time intervals and multipoint correlation functions. Using the developed formalism, we calculate the Lyapunov exponent, which determines the decay rate of correlations on a non-falling trajectory.

DOI: 10.1134/S0021364020180034

1. Balancing an inverted pendulum under a given time-dependent horizontal force $f(t)$ is a famous mathematical problem formulated by Courant and Robbins in their book "What is Mathematics?" (first edition in 1941) [1], where Whitney was credited as the author of the problem. Using fairly general mathematical arguments based on the intermediate value theorem, they showed that for any force $f(t)$ acting during a finite time interval $[0, T]$, an initial position of the pendulum in the upper half-plane can be chosen such that it will remain in the upper half-plane during the further evolution for all $t \in [0, T]$. The existence of a *non-falling trajectory* (non-FT) in the Whitney problem has been the subject of an ongoing debate in the mathematical literature [2, 3], resulting in a critical analysis and refinement of the original arguments of Courant and Robbins. Fresh interest in the problem of an inverted pendulum is associated with Arnold, whose view in 2002 was that this problem still awaits a rigorous solution [4]. In 2014, Polekhin presented a proof of the existence of the non-FT using the Wazewski topological principle [5]. This work provoked several publications generalizing his approach and proposing new topological methods [6–8] (see [8, 9] for good reviews of the history of the Whitney problem).

Recently we developed a theory of the statistical description of a *never falling trajectory* (NFT) of an inverted pendulum under the action of a random force [10]. An NFT can be regarded as the limit of non-FTs in the Whitney problem as the length T of the time interval tends to infinity. The NFT concept is illustrated in Fig. 1, which shows numerical solutions to the boundary value problem for the pendulum equation (the angle θ is measured from the vertical)

$$\ddot{\theta} = \omega^2 \sin\theta + f(t)\cos\theta \qquad (1)$$

with different initial and final values $\theta(0) = \theta_1$ and $\theta(T) = \theta_2$ and a sufficiently rapidly varying force $f(t)$. For any $\theta_{1,2}$ in the strip $-\pi/2 < \theta_{1,2} < \pi/2$, a non-falling solution $(-\pi/2 < \theta(T) < \pi/2)$ of this boundary value problem exists and is unique [10]. As θ_1 and θ_2 run through all possible values in the strip, the set of corresponding non-FTs form a bundle, shown in color in Fig. 1. This bundle shrinks as one moves away from the boundary, becoming exponentially thin in the middle of the interval for large T. In the limit $T \to \infty$, when the pendulum must be balanced on the entire real axis, the non-FT bundle for the Whitney problem on a finite time interval becomes infinitely thin and defines a unique *never falling trajectory*, which is a functional of the given force $f(t)$.

In [10], we studied the statistical properties of an NFT in the case where the driving force is Gaussian white noise with the correlator

$$\langle f(t) f(t') \rangle = 2\alpha\delta(t - t'), \qquad (2)$$

and calculated the instantaneous distribution function $P(\theta, p)$ of the angle θ and its velocity $p = \dot{\theta}$. Our approach is based on the supersymmetric field theory formulation of stochastic dynamics proposed by Parisi and Sourlas [11–13], which allows averaging over the random force at the very beginning of the calculations. It is essential that for the considered problem, the

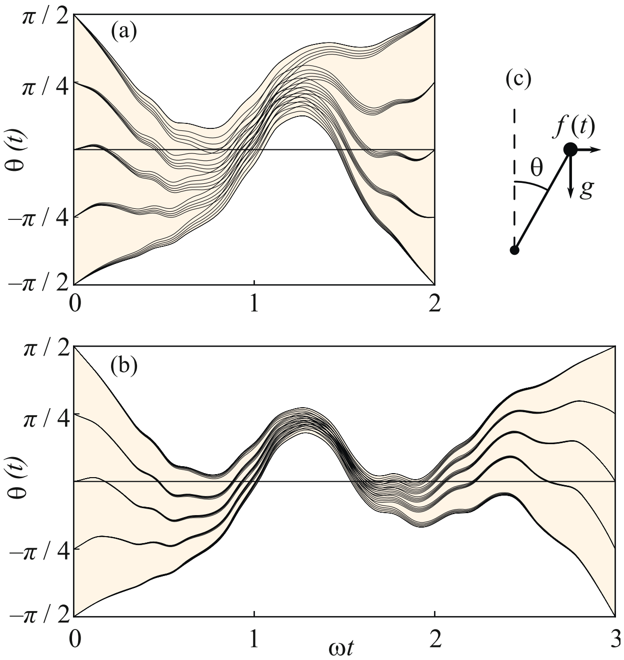

Fig. 1. (Color online) Examples of non-falling trajectories for the pendulum equation of motion (1) considered on two time intervals $T =$ (a) $2/\omega$ and (b) $3/\omega$. For any choice of θ_1 and θ_2 in the upper half-plane ($|\theta| < \pi/2$), there exists a unique non-falling solution satisfying the boundary conditions $\theta(0) = \theta_1$ and $\theta(T) = \theta_2$. We plot 25 such trajectories with $\theta_{1,2} = (-1, -0.5, 0, 0.5, 1) \times \pi/2$. The driving force is $f(t) = 4\sum_{n=1}^{40}\cos(k\omega t + k^4)$ in both cases. (c) Inverted pendulum under the action of a horizontal force.

Parisi–Sourlas method is free from the problem of the sign of the fermionic determinant because of the uniqueness of the non-FT. Using the idea of reducing the one-dimensional functional integral to an effective quantum mechanics [14], we were able to express the distribution function $P(\theta, p)$ in terms of the zero mode of the transfer-matrix Hamiltonian, which reduces to the Fokker–Planck operator with a special type of boundary conditions ensuring that the trajectories do not leave the strip.

Here, we extend the ideas of [10] and consider a range of issues related to the Lyapunov exponent for a non-FT. The Lyapunov exponent determines both the law of the convergence of a non-FT on a finite time interval to the NFT on an infinite time interval (see Fig. 1) and the decay of different-time correlators on the NFT. From the technical standpoint, our result consists in describing the entire spectrum of the transfer-matrix Hamiltonian, whose zero mode was studied in [10]. In this language, the Lyapunov exponent is determined by the energy of the first excited state. The developed theory allows calculating any correlation functions for a non-FT on infinite, semi-infinite, and finite time intervals.

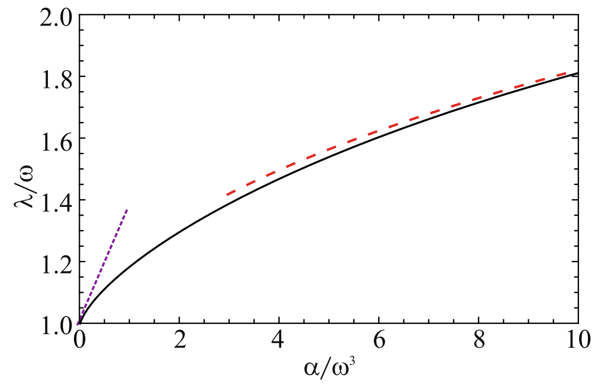

Fig. 2. (Color online) Lyapunov exponent for a non-falling trajectory versus the driving strength in units of the parameter α/ω^3: the dotted line shows the linear part of the asymptotic form for small α, and the dashed line shows the first three terms of expansion (4) for large α/ω^3.

We show that the Lyapunov exponent in the Whitney problem with white noise driving (2) can be written as

$$\lambda = \omega g(\alpha/\omega^3), \tag{3}$$

where the function $g(x)$ has the asymptotic behavior

$$g(x) = \begin{cases} 1 + \dfrac{3}{8}x - \dfrac{525}{1024}x^2 + \ldots, & x \ll 1, \\ 0.66x^{1/3} + 0.26 + 0.30x^{-1/3}\ldots, & x \gg 1. \end{cases} \tag{4}$$

In the absence of driving ($\alpha = 0$), the Lyapunov exponent $\lambda = \omega$ is determined by the exponential instability of the trajectories near the upper pendulum position. For weak driving ($\alpha/\omega^3 \ll 1$), the typical non-FT angle is of the order $\theta \sim (\alpha/\omega^3)^{1/2}$ [10], and the nonlinearity of Eq. (1) leads to an increase in the Lyapunov exponent, which can be expanded in an asymptotic series in powers of the small parameter α/ω^3. Finally, under strong driving ($\alpha/\omega^3 \gg 1$), the Lyapunov exponent reaches the limiting value $\lambda \approx 0.66\alpha^{1/3}$. We show the numerically found dependence of the Lyapunov exponent on α/ω^3 in Fig. 2.

2. According to the approach developed in [10], the statistical properties of a non-FT are expressed in terms of the two-component "wavefunction" $\hat{\Psi}(\theta, p) = (\Psi, \Phi)^T$, whose evolution is governed by the imaginary-time Schrödinger equation with the corresponding transfer-matrix Hamiltonian:

$$\frac{\partial}{\partial t}\begin{pmatrix}\Psi \\ \Phi\end{pmatrix} = -\mathcal{H}\begin{pmatrix}\Psi \\ \Phi\end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L & -1 \\ V_2 & L \end{pmatrix}, \tag{5}$$

where L is the Fokker–Planck operator for the Kramers problem [15],

$$L = p\partial_\theta + \omega^2 \sin\theta\partial_p - \alpha\cos^2\theta\partial_p^2, \tag{6}$$

and the potential V_2 has the form

$$V_2 = -\omega^2\cos\theta - \alpha\sin 2\theta\partial_p. \tag{7}$$

In [10], we studied the one-point correlation function of the NFT on the infinite time interval, which is determined by the zero mode $\hat{\Psi}_0$ of the Hamiltonian $\mathcal{H}$. Finding the zero mode is significantly simplified due to the presence of the Becchi–Rouet–Stora–Tuytin (BRST) symmetry of the action in the Parisi–Sourlas representation of stochastic dynamics [13], which allows expressing both components of $\hat{\Psi}(\theta, p)$ in terms of a scalar superpotential $\psi(\theta, p)$ via

$$\Psi = \partial_p\psi, \quad \Phi = -\partial_\theta\psi. \tag{8}$$

The time evolution of ψ is determined by the Fokker–Planck operator L:

$$\frac{\partial\psi}{\partial t} = -L\psi. \tag{9}$$

However, reduction (8) works neither for calculating multi-time correlation functions of the NFT nor for describing the statistics of a non-FT on bounded intervals. In the former case, the BRST symmetry is broken by the operators of physically observable quantities acting identically on the wavefunction components Ψ and Φ. In the latter case, the BRST symmetry is broken by the BRST-asymmetric initial condition at the boundary of the interval (see Eq. (10) below). In both cases, to describe the non-FT statistics, one must work with the two-components wavefunction (Ψ, Φ) and understand the properties of the Hamiltonian $\mathcal{H}$.

We start with discussing the initial condition for the wavefunction at the boundary of an interval. To ensure that the non-FT is unique, we must fix the value of θ at the boundary. (Generally speaking, one can fix the value of $\dot{\theta}$ or even a linear combination of θ and $\dot{\theta}$, but for simplicity, we assume that the angle is given.) By construction, the wavefunction $\hat{\Psi}$ is closely related to the partition function of the supersymmetric functional integral [10]. Right at the boundary, it cannot contain Grassmann variables, which leads to the component Φ vanishing. Hence, the wavefunction at the interval boundary with the fixed value $\theta = \theta_0$ has the form

$$\hat{\Psi}^{(b)}_{\theta_0} = \begin{pmatrix} \omega^{-1}\delta(\theta-\theta_0) \\ 0 \end{pmatrix}. \tag{10}$$

Consider the boundary value problem in the interval $[T_L, T_R]$ with the boundary conditions $\theta(t_L) = \theta_L$ and $\theta(t_R) = \theta_R$. The essence of the reduction of the Parisi–Sourlas integral to the quantum mechanics (5) is that the correlation function $\langle O_1(t_1)O_2(t_2)\ldots\rangle$ of physical quantities O_i at the instants t_i $(t_1 < t_2 < \ldots)$ can be represented as the matrix element

$$\langle\hat{\Psi}^{(b)}_{\theta_R}|\ldots O_2(t_2)e^{-\mathcal{H}(t_2-t_1)}O_1(t_1)e^{-\mathcal{H}(t_1-t_L)}|\hat{\Psi}^{(b)}_{\theta_L}\rangle, \tag{11}$$

where the scalar product of two wavefunctions is defined as [10]

$$\langle\hat{\Psi}|\hat{\Psi}'\rangle = \int d\theta dp[\Psi(\theta,p)\Phi'(\theta,-p) + \Phi(\theta,p)\Psi'(\theta,-p)]. \tag{12}$$

In [10], we studied the instantaneous joint distribution function $P(\theta, p)$ of the angle and velocity on the NFT corresponding to the operator $O = \delta(\theta-\theta_0)\delta(p-p_0)$. Replacing $\hat{\Psi}_{L,R}$ with the zero mode and using Eq. (8), one can express $P(\theta, p)$ in terms of the Poisson bracket of the superpotential ψ:

$$P(\theta, p) = \{\psi(\theta,p), \psi(\theta,-p)\}_{\theta,p}. \tag{13}$$

Both the Hamiltonian in Eq. (5) and the Fokker–Planck operator (6) are non-Hermitian. Generally speaking, such operators can lack a complete system of eigenfunctions. However, it is known that in the presence of friction the Fokker–Planck operator can be diagonalized [15], which makes it possible to construct a system of biorthogonal eigenfunctions and work with them practically as with eigenfunctions of a Hermitian operator [16]. However, there is no friction in our case, and we should therefore expect that the operators $\mathcal{H}$ and L reduce to the Jordan normal form. This results not in a simple exponential decay of correlators as $t \to \infty$ but in the appearance of additional powers of time (e.g., as seen in Eq. (30)).

3. To illustrate the developed approach, we consider the case of a *weak noise* $(\alpha/\omega^3 \ll 1)$ in detail, where the Jordan structure of the operators $\mathcal{H}$ and L can be studied analytically. We start with the Fokker–Planck operator. In the considered limit, the deviation of the pendulum from the vertical is small $(\theta \ll 1)$, and the operator (6) can be replaced with

$$L = p\partial_\theta + \omega^2\theta\partial_p - \alpha\partial_p^2. \tag{14}$$

The zero mode of this operator corresponding to the NFT has the form

$$\psi_0(\theta, p) = \mathrm{erf}(z)/2, \tag{15}$$

where we introduce "holomorphic" and "antiholomorphic" coordinates with different signs of the momentum,

$$z = \kappa(p - \omega\theta), \quad \bar{z} = -\kappa(p + \omega\theta), \tag{16}$$

where $\kappa = \sqrt{\omega/2\alpha}$. The spectrum of the operator (14) can be found using the identity $[L, \partial_z] = \omega\partial_z$, which allows generating the eigenfunctions by consecutively differentiating the zero mode with respect to z. We thus find the eigenfunction of the nth excited state $(n = 1, 2, 3, \ldots)$ with the energy $\epsilon_n = n\omega$:

$$\psi_n = \frac{1}{\sqrt{\pi}} H_{n-1}(z) e^{-z^2}, \tag{17}$$

where $H_n(z) = (-1)^n e^{z^2} d^n e^{-z^2}/dz^n$ is the Hermite polynomial (in the physical definition). However, the functions $\psi_n(\theta, p)$ thus constructed depend only on the difference $p - \omega\theta$ (do not contain $\bar{z}$) and therefore do not form a complete basis. This circumstance is related to the fact that the non-Hermitian operator (6) can be brought to the Jordan normal form and in addition to the eigenfunction has several generalized eigenfunction corresponding to the same eigenvalue ε_n. It is easy to verify that the eigenfunction ψ_n has $n - 1$ generalized eigenfunctions, which we choose in the form

$$\psi_{n,k} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} H_k(\bar{z}) H_{n-k-1}(z) e^{-z^2}, \tag{18}$$

where the index k ranges from 1 to $n - 1$. Together with $\psi_{n,0} = \psi_n$, they form the basis of a Jordan block of dimension n corresponding to the energy $\epsilon_n = n\omega$:

$$L\psi_{n,k} = \epsilon_n \psi_{n,k} + \omega \psi_{n,k-1} \tag{19}$$

(to truncate the chain at the eigenfunction $\psi_{n,0}$, we set $\psi_{n,-1} = 0$).

The constructed system of functions is complete. An arbitrary function can be expanded in the basis $\psi_{n,k}$ using the orthogonality relation

$$\langle \psi_{n,k} | \psi_{n',k'} \rangle_z = (-1)^{n-1} \delta_{n,n'} \delta_{k+k'+1,n}, \tag{20}$$

where the scalar product $\langle \cdot | \cdot \rangle_z$ is defined as

$$\langle \psi | \psi' \rangle_z = \int dz d\bar{z}\, \psi(\bar{z}, z) \psi'(z, \bar{z}), \tag{21}$$

and exchanging the arguments in one of the functions thus agrees with the sign change for p in Eq. (12). We note that the integration measures in Eqs. (12) and (21) are related by $dz d\bar{z} = 2\omega\kappa^2 d\theta dp$.

During the evolution of the wavefunction $\psi_{n,k}$ under the action of the operator L, other states of the Jordan block corresponding to the same energy are mixed into it, which leads to the appearance of powers of t on top of the exponential decay:

$$e^{-Lt} \psi_{n,k} = e^{-n\omega t} \sum_{m=0}^{k} \frac{(-\omega t)^m}{m!} \psi_{n,k-m}. \tag{22}$$

We now turn to studying the spectral properties of the Hamiltonian $\mathcal{H}$ in Eq. (5). In the considered case of weak noise, Eq. (7) gives $V_2 = -\omega^2$, which partitions the state space of $\mathcal{H}$ into even and odd sectors with the wavefunctions $\hat{\Psi}_{e,o} = (\Psi, \pm\omega\Psi)^T$ evolving independently with the Hamiltonians $\mathcal{H}_{e,o} = L \mp \omega$. The system of eigenfunctions and generalized eigenfunctions of the operator L constructed above thus allows completely describing the evolution of the doublet $\hat{\Psi}$ under the action of the Hamiltonian $\mathcal{H}$.

Consider the evolution of the wavefunction (10) away from the boundary in the limit $\alpha/\omega^3 \ll 1$. Decomposing it into even and odd components, we obtain

$$e^{-\mathcal{H}t} \hat{\Psi}^{(\mathrm{b})}_{\theta_0} = \begin{pmatrix} \omega^{-1} \cosh \omega t \\ \sinh \omega t \end{pmatrix} e^{-Lt} \delta(\theta - \theta_0). \tag{23}$$

To calculate the evolution of the delta function, we expand it in the basis $\psi_{n,k}$:

$$\delta(\theta - \theta_0) = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} c_{n,k} \psi_{n,k}. \tag{24}$$

The coefficients $c_{n,k}$ can be obtained using orthogonality relations (20) and the properties of Hermite polynomials

$$H_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} (2y)^{n-k} H_k(x) \tag{25}$$

following from the Taylor expansion, and are given by

$$c_{n,k} = (-1)^{n-1} 2\kappa\omega \frac{(2\kappa\omega\theta_0)^{n-2k-1}}{(n - 2k - 1)!}. \tag{26}$$

The evolution of the delta function in Eq. (23) follows from expansion (24) and relations (22). The memory of the boundary is lost in the characteristic time ω^{-1} (the inverse Lyapunov exponent). During this time, the difference between the two components of the wavefunction $\hat{\Psi}$ is lost, and they both take the value determined by the state $\psi_{1,0}$ with the minimum energy $\epsilon_1 = \omega$:

$$\lim_{t \to \infty} e^{-\mathcal{H}t} \hat{\Psi}^{(\mathrm{b})}_{\theta_0} = \hat{\Psi}_0 = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \kappa \psi_{1,0}, \tag{27}$$

which is just the zero mode of (5) in the limit $\alpha/\omega^3 \ll 1$.

4. We show how the developed spectral theory of the operators $\mathcal{H}$ and L allows systematically calculating various correlation functions of the non-FT *in the case of weak noise*. The results in this section can also be obtained directly by using the explicit expression for the non-FT in terms of $f(t)$ with subsequent averaging over Gaussian white noise (2) [10], but deriving them using the transfer-matrix formalism is important methodologically because it illustrates the general scheme and allows verifying its workability.

We begin by considering the calculation of the pair correlator for the NFT angle on the entire real axis. Substituting the zero mode (27) into the general formula (11) and taking into account that only the even sector of the theory does contribute, we can express the correlator in terms of the scalar product (21) in the z-representation as

$$\langle\theta(0)\theta(t)\rangle = \langle\psi_{1,0}|\theta e^{-(L-\omega)t}\theta|\psi_{1,0}\rangle_z. \tag{28}$$

Using Eqs. (16) and (18), we can express $\theta\psi_{1,0}$ in terms of the functions $\psi_{2,0}$ and $\psi_{2,1}$. Then using the evolution law (22), we obtain

$$e^{-(L-\omega)t}\theta\psi_{1,0} = e^{-\omega t}\,\frac{\psi_{2,1} - (1/2 + \omega t)\psi_{2,0}}{2\kappa\omega}. \tag{29}$$

Calculating the matrix element (28) as the overlap between the states $e^{-(L-\omega)t}\theta\psi_{1,0}$ and $\theta\psi_{1,0}$ with the help of relations (20), we find the sought pair correlator:

$$\langle\theta(0)\theta(t)\rangle = \langle\theta^2\rangle(1 + \omega t)e^{-\omega t}, \tag{30}$$

where, as obtained in [10],

$$\langle\theta^2\rangle = \alpha/2\omega^3. \tag{31}$$

The appearance on the background $e^{-\omega t}$ of a contribution linearly increasing with time is related to excitation of the states $\psi_{2,0}$ and $\psi_{2,1}$ corresponding to the Jordan block of dimension 2.

In a similar way one can calculate more complicated correlators of the NFT. For example,

$$\langle\theta^2(0)\theta^2(t)\rangle = \langle\theta^2\rangle^2\left[1 + 2(1 + \omega t)^2 e^{-2\omega t}\right]. \tag{32}$$

Formally, the operator θ^2 here applied to $\psi_{1,0}$ excites the Jordan triplet $\psi_{3,0}$, $\psi_{3,1}$, $\psi_{3,2}$, which leads to the appearance of terms up to t^2 on the background of the exponential decay. But the structure of the correlator (32) is related to the Gaussian statistics of θ on the NFT [10], which allows expressing it in terms of pair correlator (30) using the Wick theorem. Generalizing the developed formalism to multipoint correlators is also straightforward.

As the next example, we consider the calculation of the average angle $\langle\theta(t)\rangle_{\theta_0}$ for the non-FT on the semi-infinite time interval $t > 0$ with the boundary condition $\theta(0) = \theta_0$. According to Eq. (11), the average angle is given by the matrix element $\langle\theta(t)\rangle_{\theta_0} = \langle\hat\Psi_0|\theta e^{-\mathcal{H}t}|\hat\Psi^{(b)}_{\theta_0}\rangle$. It is easiest to calculate by convoluting Eq. (29) with the wavefunction (10) at the boundary. Integrating over the momentum, we see that the contribution from $\psi_{2,0} = 2ze^{-z^2}/\sqrt{\pi}$ disappears because it is odd in z, and we obtain the simple exponential decay

$$\langle\theta(t)\rangle_{\theta_0} = \theta_0 e^{-\omega t}. \tag{33}$$

One can derive the same expression differently by calculating the matrix element θ between the zero mode $\psi_{1,0}$ and evolved boundary wavefunction (23). Such matrix elements are nonzero only with the Jordan doublet $\psi_{2,0}$ and $\psi_{2,1}$. However, according to Eq. (26), $\psi_{2,1}$ is not included in the expansion of the delta function, while $\psi_{2,0}$ is an eigenfunction and does not generate a linear term during evolution. As a result, we again come to Eq. (33).

The comparison of Eqs. (30) and (33) shows that despite the presence of the additional factor ωt in Eq. (30), the Lyapunov exponent can be standardly determined from either of the correlators at large times:

$$\lambda = -\lim_{t\to\infty}\frac{\partial\ln\langle\theta(0)\theta(t)\rangle}{\partial t} = -\lim_{t\to\infty}\frac{\partial\ln\langle\theta(t)\rangle_{\theta_0}}{\partial t}. \tag{34}$$

5. We now proceed to calculating the Lyapunov exponent for the non-FT *for arbitrary values of the parameter* α/ω^3. The Lyapunov exponent, which governs the decay of the correlations at large times, is determined by the energy of the first excited state. As shown above, in the case of weak driving, $\lambda = \omega$. As the parameter α/ω^3 increases, the anharmonicity of the pendulum leads to a deviation of λ from ω.

For a small value of the parameter $\alpha/\omega^3 \ll 1$, the nonlinear terms in Eq. (6) can be taken into account perturbatively, which allows obtaining both a correction to the eigenfunction ψ_n, which becomes dependent on the antiholomorphic coordinate $\bar z$, and a correction to the eigenvalue ϵ_n. This procedure looks especially simple for the first excited state, which is nondegenerate and has no generalized eigenfunctions. For this, we represent the eigenfunction and the corresponding energy as power series in the small parameter $x = \alpha/\omega^3$:

$$\psi_1 = [1 + h_1(z,\bar z)x + h_2(z,\bar z)x^2 + \ldots]e^{-z^2},$$
$$\epsilon_1 = \omega(1 + \gamma_1 x + \gamma_2 x^2 + \ldots),$$

where $h_m(z,\bar z)$ is a polynomial of a degree not exceeding $4m$. Substituting these expressions in the equation $L\psi_1 = \epsilon_1\psi_1$ and solving sequentially in each order in x, we can calculate the first few polynomials $h_m(z,\bar z)$ and the coefficients γ_m. The result for ϵ_1 defining the Lyapunov exponent is given in Eq. (4).

A similar approach allows also finding corrections to the zero mode (15) of the superpotential ψ_0 in powers of α/ω^3. As anticipated from the supersymmetry of the theory, its energy remains zero. The found corrections allow obtaining an analytic expansion for the one-point statistics of the NFT, calculated numerically in [10]. In particular, they allow refining formula (31) for $\langle\theta^2\rangle$,

$$\langle\theta^2\rangle = \frac{x}{2} - \frac{13}{16}x^2 + \frac{26989}{12288}x^3 + \ldots, \tag{35}$$

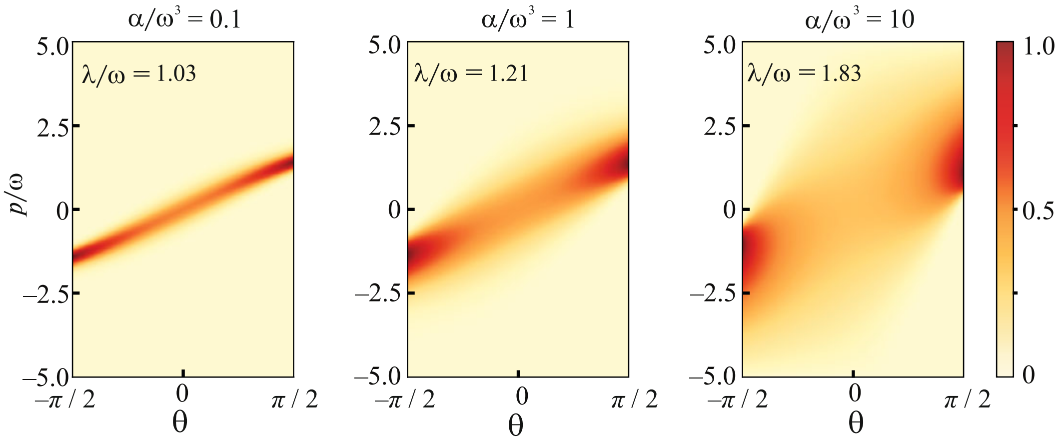

Fig. 3. (Color online) First excited state $\psi_1(\theta, p)$ of the operator (6) for three values of the parameter $\alpha/\omega^3 = 0.1, 1, 10$. The wavefunction is normalized to the maximum value.

and describing the non-Gaussianity of the distribution function $P(\theta)$ characterized by the fourth cumulant $\langle\langle\theta^4\rangle\rangle = \langle\theta^4\rangle - 3\langle\theta^2\rangle^2$:

$$\langle\langle\theta^4\rangle\rangle = -\frac{241}{256}x^3 + \frac{64725}{8192}x^4 + \ldots. \quad (36)$$

Note that the difference from the normal distribution measured by the kurtosis $\langle\langle\theta^4\rangle\rangle/\langle\theta^2\rangle^2$ occurs only in the first order in $x = \alpha/\omega^3$. A negative value of $\langle\langle\theta^4\rangle\rangle$ is related to suppression of the tails of $P(\theta)$ due to the finiteness of the interval $(-\pi/2, \pi/2)$.

In the case of an arbitrary noise strength, the excited states of operator (6) can be constructed only numerically. To determine the Lyapunov exponent $\lambda = \epsilon_1$, we must find the first excited state by solving the equation $L\psi = \epsilon_1\psi$ with the boundary conditions

$$\psi(\pi/2, p < 0) = \psi(\theta, -\infty) = 0, \quad (37a)$$

$$\psi(-\pi/2, p > 0) = \psi(\theta, \infty) = 0. \quad (37b)$$

These boundary conditions are similar to the boundary conditions for the zero mode of the superpotential derived in [10], with the only difference that in the part of the boundary where the wavefunction is specified, its value is zero and not $\pm 1/2$.

In Fig. 3, we show the first excited state determined numerically for various values of the parameter α/ω^3. For small α/ω^3, the function $\psi_1(\theta, p)$ is close to the Gaussian $\psi_{1,0}(z)$, slightly increasing near $\theta = \pm\pi/2$. As α/ω^3 increases, the maximum of $\psi_1(\theta, p)$ near the boundaries of the interval become more pronounced, and at $\alpha/\omega^3 \to \infty$, the first mode has two humps localized near the boundaries. In Fig. 2, we plot the energy of the first mode (which determines the Lyapunov exponent) as a function of the parameter α/ω^3. For small α/ω^3, the numerical calculation agrees with Eq. (4) obtained using the perturbation theory up to the values $\alpha/\omega^3 \approx 0.25$. For larger α/ω^3, the Lyapunov exponent in units of ω can be expanded in powers of $(\alpha/\omega^3)^{-1/3}$ with the leading term $\lambda \approx 0.66\alpha^{1/3}$.

In conclusion, we note that the developed theory is a generalization of the supersymmetric approach proposed in [10] to the case of a non-FT on finite time intervals and to multipoint correlation functions. The suggested classification of the excited states of the transfer-matrix Hamiltonian completes the construction of the theory of the statistical properties of a non-FT in the Whitney problem with random short-range driving. The developed formalism allows finding any correlation functions on a non-FT by solving partial differential equations of the Fokker–Planck type with specific boundary conditions.

ACKNOWLEDGMENTS

We are grateful to A.V. Khvalyuk and I.V. Poboiko for the help with numerical calculations.

FUNDING

This work was supported by the Russian Science Foundation (project no. 20-12-00361).

REFERENCES

1. R. Courant and H. Robbins, *What is Mathematics? An Elementary Approach to Ideas and Methods* (Oxford Univ. Press, New York, 1996).
2. A. Broman, Nordisk Mat. Tidskrift **6**, 78 (1958).
3. T. Poston, Manifold **18**, 6 (1976).

4. V. Arnold, *What is Mathematics?* (MCCME, Moscow, 2002) [in Russian].
5. I. Yu. Polekhin, Nelin. Dinam. **10**, 465 (2014); arXiv: 1407.4787.
6. O. Zubelevich, Appl. Math. (Warsaw) **42**, 159 (2015).
7. S. V. Bolotin and V. V. Kozlov, Izv. Math. **79**, 894 (2015).
8. R. Srzednicki, Discrete Contin. Dyn. Syst. Ser. S **12**, 2127 (2019).
9. A. Shen, arXiv: 1907.01598.
10. N. A. Stepanov and M. A. Skvortsov, arXiv: 2006.13819.
11. G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).
12. G. Parisi and N. Sourlas, Nucl. Phys. B **206**, 321 (1982).
13. J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 2015), Chap. 16.
14. K. B. Efetov and A. I. Larkin, Sov. Phys. JETP **58**, 444 (1983).
15. H. Risken, *The Fokker–Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1996).
16. G. E. Shilov, *Linear Algebra* (Dover, New York, 1977).