Magnetic disorder in superconductors: Enhancement by mesoscopic fluctuations

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We study the density of states (DOS) and the transition temperature T_c in a dirty superconducting film with rare classical magnetic impurities of an arbitrary strength described by the Poissonian statistics. We take into account that the potential disorder is a source for mesoscopic fluctuations of the local DOS, and, consequently, for the effective strength of magnetic impurities. We find that these mesoscopic fluctuations result in a non-zero DOS for all energies in the region of the phase diagram where without this effect the DOS is zero within the standard mean-field theory. This mechanism can be more efficient in filling the mean-field superconducting gap than rare fluctuations of the potential disorder (instantons). Depending on the magnetic impurity strength, the suppression of T_c by spin-flip scattering can be faster or slower than in the standard mean-field theory.

Introduction. — The properties of superconductors in the presence of impurities have remained at the focus of intense theoretical and experimental research during the past half-century. It is generally accepted that the potential scattering in s-wave superconductors affects neither the transition temperature, T_c, nor the density of states (DOS), $\rho(E)$. This statement usually referred to as Anderson's theorem [1–3] is valid for sufficiently good metals. As the potential disorder increases, the emergent inhomogeneity due to the interplay of quantum interference (Anderson localization) and interaction leads to modification of T_c [4–12] and $\rho(E)$ [13–16], with the effect being controlled by the parameter $1/(k_F l) \ll 1$ (where k_F is the Fermi momentum and l is the mean free path).

Magnetic impurities violating the time-reversal symmetry affect superconductivity much stronger, already at $k_F l \to \infty$. Classical magnetic impurities lead both to suppression of T_c and to reduction of the superconducting gap in $\rho(E)$ with the increase of their concentration n_s [17]. Beyond the Born limit, magnetic impurities produce degenerate subgap bound states (see Fig. 1a). Their hybridization results in the formation of an energy band giving rise to a nontrivial DOS structure [18–21]. The account for the Kondo effect [22–24], the indirect exchange interaction between magnetic impurities [25], or the spin-flip scattering assisted by the electron-phonon interaction [26] can lead to the reentrant behavior of T_c vs. n_s (see Ref. [27] for a review).

A hard gap in $\rho(E)$ obtained for superconductors with magnetic impurities in the mean-field approximation is smeared by inhomogeneity. This can be due to rare fluctuations of a potential disorder [28–31], n_s [32], or superconducting order parameter [33]. A combined theory of these mechanisms has been developed in Refs. [34, 35].

In this Letter we describe a novel mechanism for smearing of the superconducting gap. We reconsider the problem of rare classical magnetic impurities with the Poissonian statistics in a dirty superconductor. The key point that distinguishes our work from the previous ones is that we take into account mesoscopic fluctuations of the local DOS in a potential disorder. Physically, this implies that the energies of subgap bound states become dependent on the spatial positions of magnetic impurities (see Fig. 1b). Averaging over these bound states results in a nonzero homogenous DOS at all energies in the region of the phase diagram where in the absence of this effect $\rho(E)$ is zero within the mean-field approximation. Motivated by the recent experiment on magnetic Gd impurities in superconducting MoGe films [36], in this Letter we develop the theory of the enhancement of magnetic disorder by mesoscopic fluctuations in the case of a dirty superconducting film.

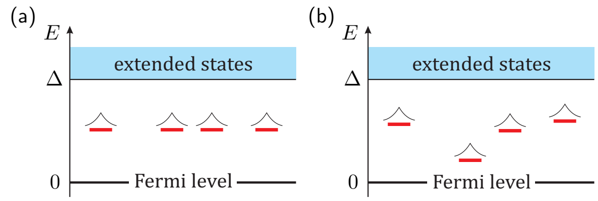

FIG. 1. (Color online) Subgap states localized at individual magnetic impurities. (a) In a clean system, the energies of all bound states are equal. (b) Mesoscopic fluctuations lead to the log-normal distribution of impurity strength [cf. Eq. (4)], rendering the energies of bound states position-dependent.

Model. — We consider a two-dimensional (2D) dirty s-wave superconductor in the presence of both potential (spin-preserving) and magnetic disorder. Scattering off the former is responsible for the dominant contribution to the momentum relaxation rate $1/\tau$. Much weaker spin-flip scattering rate is related with the exchange interaction between magnetic impurities and electrons described by the Hamiltonian

$$H_{\rm mag} = J \sum_j \psi^\dagger(\boldsymbol{r}_j) \boldsymbol{S}_j \boldsymbol{\sigma} \psi(\boldsymbol{r}_j). \qquad (1)$$

We shall treat rear magnetic disorder under standard assumptions [18–21, 31]: (i) impurity positions $\boldsymbol{r}_j$ have the

Poisson distribution; (ii) impurity spins $\boldsymbol{S}_j$ are classical statistically independent vectors with the flat distribution over their orientations, $\prod_j \delta(\boldsymbol{S}_j^2 - S^2)$.

Results. — In the mean-field approximation, a dirty superconductor in the diffusive regime is described by two coupled equations: the self-consistency equation for the superconducting order parameter Δ and the Usadel equation for the quasiclassical Green's function [37–39]. These equations can be derived as the saddle-point equations of the nonlinear sigma model. The latter provides full description of quantum effects for a dirty superconductor in the diffusive regime (see Supplemental Material [40] for details). These effects (weak localization and Aronov-Altshuler-type corrections) are responsible for the renormalization of system's parameters. In the 2D case, the magnitude of quantum corrections at the energy scale ε is governed by the parameter

$$t(\varepsilon) = \frac{1}{\pi g} \ln \frac{1}{|\varepsilon|\tau}, \tag{2}$$

where $g = h/(e^2 R_\square) \gg 1$ is the bare dimensionless conductance of the film. In a superconductor, renormalization stops at $\varepsilon \sim \max\{T_c, |\Delta|\} \sim T_c$. Assuming that the transition temperature is not too low, $t(T_c) \ll 1$, one can neglect the renormalization of the conductance and interaction parameters between the energy scales $1/\tau$ and T_c (see Refs. [41, 42] for a review). In contrast, renormalization of the magnetic-impurity part of the nonlinear sigma model is essential.

Treating this renormalization in the one-loop approximation, we obtain a modified Usadel equation. In the angular parametrization, it reads [40]:

$$\varepsilon \sin\theta_\varepsilon - \Delta\cos\theta_\varepsilon + \frac{n_s}{\pi\nu}\left\langle \frac{\mathsf{a}\sin 2\theta_\varepsilon}{1+\mathsf{a}^2+2\mathsf{a}\cos 2\theta_\varepsilon}\right\rangle_{\mathsf{a}} = 0, \tag{3}$$

where θ_ε is the energy-dependent spectral angle, ν is the normal DOS at the Fermi energy per one spin projection, and $\varepsilon = \pi T(2n+1)$ denotes the fermionic Matsubara frequencies. The averaging $\langle \dots \rangle_{\mathsf{a}}$ in Eq. (3) is defined with respect to the following log-normal distribution function:

$$\mathcal{P}_\alpha(\mathsf{a}, t) = \frac{1}{4\mathsf{a}\sqrt{\pi t}} \exp\left[-\frac{1}{4t}\left(\frac{1}{2}\ln\frac{\mathsf{a}}{\alpha} + t\right)^2\right]. \tag{4}$$

Here $\alpha = (\pi\nu JS)^2$ stands for the bare dimensionless impurity strength. Since $\mathcal{P}_\alpha(\mathsf{a}, t\to 0) \to \delta(\mathsf{a}-\alpha)$, Eq. (3) at $t = 0$ coincides with the standard Usadel equation in the case of magnetic impurities [17, 20, 21, 31]. The linearity of Eq. (3) in n_s is justified for small concentration of magnetic impurities: $n_s\xi^2/g \ll 1$, where $\xi = l/\sqrt{T_c\tau}$ is the dirty superconducting coherence length.

The quantity a in Eq. (3) plays a role of the renormalized impurity strength. Since the bare impurity strength α is proportional to the local DOS which is subjected to mesoscopic fluctuations, $\mathcal{P}_\alpha(\mathsf{a}, t)$ reflects the log-normal distribution of the local DOS in 2D weakly disordered systems [43, 44]. Contrary to naive expectations, one should average over a the Usadel equation rather than physical observables, e.g. the DOS. This is a consequence of the Poisson distribution of impurity positions $\boldsymbol{r}_j$.

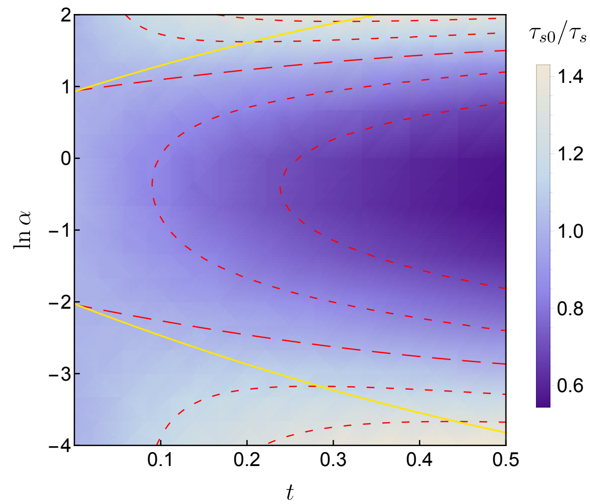

FIG. 2. (Color online) The color plot for τ_{s0}/τ_s vs. $\ln\alpha$ and t. The red dashed curves indicate isolines 0.7, 0.85, 1 (long dash), 1.15, 1.3. The yellow curves mark the position of the maximum of τ_{s0}/τ_s as a function of t for a fixed value of α.

Effective spin-flip rate. — In the vicinity of the thermal transition, $\Delta \to 0$ and we can linearize Eq. (3) with respect to θ_ε. This procedure yields $\theta_\varepsilon \approx \Delta/(\varepsilon + 1/\tau_s)$ where the effective spin-flip rate is given by

$$\frac{1}{\tau_s} = \frac{2n_s}{\pi\nu}\left\langle \frac{\mathsf{a}}{(1+\mathsf{a})^2}\right\rangle_{\mathsf{a}}. \tag{5}$$

At $t = 0$, one recoves the standard expression for the bare spin-flip rate due to magnetic impurities, $1/\tau_{s0} = 2\alpha n_s/[\pi\nu(1+\alpha)^2]$ [21]. In the limiting cases $\alpha \to 0$ and $\alpha \to \infty$, the spin-flip rate (5) becomes enhanced in comparison with the bare one: $1/\tau_s = \exp(2t)/\tau_{s0}$ and $1/\tau_s = \exp(6t)/\tau_{s0}$, respectively. For an arbitrary value of α, the asymptotic expansion at $t \ll 1$ has the form:

$$\frac{\tau_{s0}}{\tau_s} \approx 1 + \frac{2 - 16\alpha + 6\alpha^2}{(1+\alpha)^2} t(T_c) + O(t^2). \tag{6}$$

At small t, the spin-flip rate is suppressed (enhanced) for $\alpha_0 < \alpha < 1/(3\alpha_0)$ (otherwise), where $\alpha_0 = 1/(4+\sqrt{13}) \approx 0.13$. The overall behavior of the ratio τ_{s0}/τ_s as a function of t and α is illustrated in Fig. 2.

Transition temperature. — Since the spin-flip rate (5) is the only characteristic of magnetic disorder that enters the linearized solution for θ_ε, we obtain the standard equation for the superconducting transition temperature:

$$\ln\frac{T_{c0}}{T_c} = \psi\left(\frac{1}{2} + \frac{1}{2\pi T_c\tau_s}\right) - \psi\left(\frac{1}{2}\right), \tag{7}$$

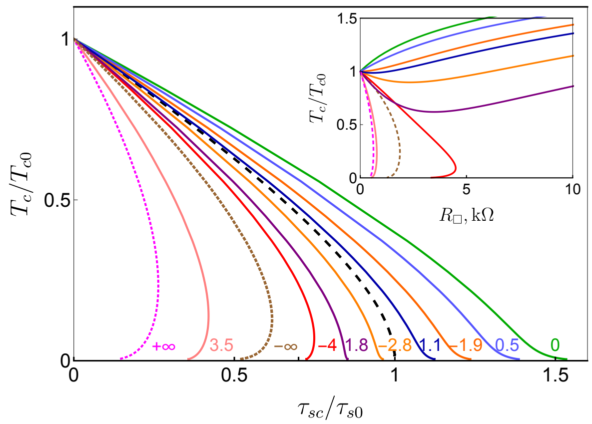

FIG. 3. (Color online) The dependence of T_c/T_{c0} on the bare spin-flip rate $1/\tau_{s0}$ for some values of the bare impurity strength α, and $R_\square = 2.5\,\mathrm{k}\Omega$ ($g = 10$). The black dotted curve, $T_c^{\mathrm{AG}}(1/\tau_{s0})$, is the solution of Eq. (7) without renormalization. Inset: The dependence of T_c/T_c^{AG} on $R_\square$ for the same values of α, and $\tau_{sc}/\tau_{s0} = 0.7$. We use $\ln(T_{c0}\tau) = 5$.

where T_{c0} denotes the transition temperature in the absence of magnetic impurities, and $\psi(z)$ stands for the digamma function. Equation (7) was derived by Abrikosov and Gor'kov in the Born limit ($\alpha \to 0$) [17], and later was shown to describe the suppression of T_c for arbitrary values of α [21]. For a scale-independent spin-flip time, $\tau_s = \tau_{s0}$, Eq. (7) defines a universal function $T_c^{\mathrm{AG}}(1/\tau_{s0})$ shown by the black dashed line in Fig. 3. Superconductivity is eventually destroyed at the critical spin-flip rate $1/\tau_{sc} = 2\pi e^{\psi(1/2)} T_{c0} \approx 0.88\, T_{c0}$ [17]. This standard approach corresponds to the limit $t = 0$, when mesoscopic fluctuations can be neglected.

An essential modification introduced by the log-normal distribution of the impurity strength (4) is that now the spin-flip rate $1/\tau_s$ depends on the parameter $t(T_c)$, i.e. on the conductance g and the transition temperature T_c itself. This leads to an unusual behavior illustrated in Fig. 3, where we present the numerical solutions of Eq. (7) for fixed values of g and $T_{c0}\tau$ and for various values of α. At finite t, dependence of $1/\tau_s$ on T_c renders the curves $T_c(1/\tau_{s0})$ sensitive to a particular value of α. In the range $\alpha_0 < \alpha < 1/(3\alpha_0)$, the spin-flip rate decreases monotonously down to zero with increasing t. Therefore the reduction of T_c with the increase of $1/\tau_{s0}$ is slower than for $t = 0$. This agrees qualitatively with the slowdown of T_c suppression with increasing the film resistance measured in Ref. [36]. In the opposite case, for $\alpha < \alpha_0$ and $\alpha > 1/(3\alpha_0)$, the dependence of T_c on $1/\tau_{s0}$ is qualitatively different since the ratio τ_{s0}/τ_s can be larger than unity and is a non-monotonous function of t. Since the spin-flip rate is enhanced, the reduction of T_c with the increase of $1/\tau_{s0}$ is faster than in the case $t = 0$. The non-monotonicity of τ_{s0}/τ_s results in the existence of two solutions of Eq. (7) for T_c. Formally, Eq. (7) admits the solution with nonzero T_c for any value of the parameter τ_{sc}/τ_{s0}. However, we remind that our approach is valid provided the inequality $T_c \gg \exp(-\pi g)/\tau$ holds.

The dependence of the spin-flip rate on g transforms into the dependence of T_c on the film conductance. To illustrate this effect, we fix the value of the parameter τ_{sc}/τ_{s0} and plot the ratio $T_c/T_c^{\mathrm{AG}}(1/\tau_{s0})$ on the film resistance $R_\square$ for some values of α in the inset to Fig. 3. Since for $\alpha_0 < \alpha < 1/(3\alpha_0)$ the spin-flip rate decreases monotonously with the increase of t, T_c is enhanced with respect to T_c^{AG}. The non-monotonous dependence of $1/\tau_s$ on t obtained for $\alpha < \alpha_0$ and $\alpha > 1/(3\alpha_0)$ leads to the reentrant behavior of T_c on $R_\square$.

It is worthwhile to mention that not only the suppression of T_c by magnetic impurities but also the reduction of Δ is modified at finite g due to the log-normal distribution of the effective impurity strength [45].

Density of states. — Consider now the superconducting phase with a finite Δ. The DOS can be obtained from the solution of Eq. (3) after analytic continuation to real energies E: $\rho(E) = 2\nu\,\mathrm{Re}\cos\theta_{-iE+0}$. It is convenient to parametrize the spectral angle as $\theta = \pi/2 + i\psi$. Without renormalization ($t = 0$), the angle $\psi(E)$ should be determined from equation $F_E(\psi) = 0$, where [17, 20, 21, 31]

$$F_E(\psi) = \sinh\psi - \frac{E}{\Delta}\cosh\psi - \frac{[\alpha n_s/(\pi\nu\Delta)]\sinh 2\psi}{1 + \alpha^2 - 2\alpha\cosh 2\psi}. \quad (8)$$

This leads to a complicated structure of the DOS at energies $|E| < \Delta$, which depends on the values of α and n_s (see Ref. [35] for a review). In the case $\tau_{s0}\Delta < (\frac{1+\alpha}{1-\alpha})^2$, the impurity band touches the Fermi energy, leading to a finite DOS at $E = 0$. In what follows, we shall consider the opposite regime, $\tau_{s0}\Delta > (\frac{1+\alpha}{1-\alpha})^2$, in which $\rho(E)$ has a finite gap E_g for $t = 0$. The gap opens since $F_E(\psi) = 0$ possesses only real solutions at energies $|E| < E_g$.

The main effect of mesoscopic fluctuations on magnetic disorder is the smearing of the hard gap E_g and the appearance of a finite DOS for all energies. Typical modification of the DOS at finite t is illustrated in Fig. 4, where we plot $\rho(E)$ obtained by numerical solution of Eq. (3) at $\varepsilon \to -iE + 0$. With the increase of t, the states are redistributed in energy, providing a rapidly growing number of subgap states with $E < E_g$.

In order to study $\rho(E)$ in the subgap region analytically, we split ψ into the real and imaginary parts, $\psi = \psi' + i\psi''$. For $t \ll 1$, the DOS at $|E| < E_g$ is small, such that $\psi'' \ll 1$. Then the integral over a in Eq. (3) is dominated by vicinities of two points $\exp(\pm 2\psi')$. Integrating over a, we obtain the following DOS for $t \ll 1$ [40]:

$$\rho(E) = \frac{n_s \cosh\psi'}{\Delta\,\partial_{\psi'} F_E(\psi')} \sum_{\sigma=\pm 1} e^{2\psi'\sigma} \mathcal{P}_\alpha(e^{2\psi'\sigma}, t)\,. \quad (9)$$

For $t \ll 1$, one can find ψ' from the approximate relation $F_E(\psi') = 0$. Then Eqs. (8) and (9) determine the subgap DOS in the parametric form.

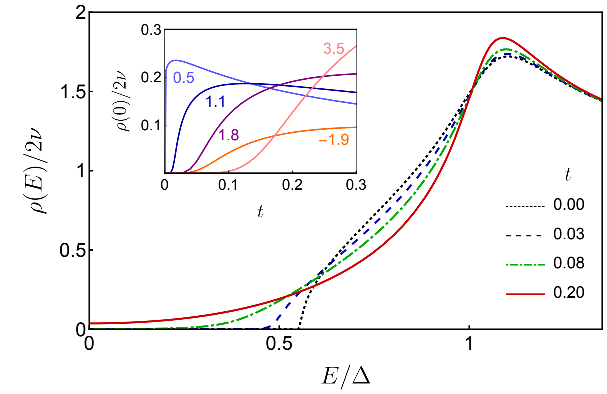

FIG. 4. (Color online) The energy dependence of the DOS for some values of the parameter t, and $\alpha = 0.05$. Inset: The DOS at the Fermi energy as a function of t for some values of the bare impurity strength α. We use $1/(\tau_{s0}\Delta) = 0.1$.

A profound feature of the DOS is its finite value right at the Fermi energy. At zero energy, $\psi' = 0$ is a root of the function (8). Then using Eq. (9) we find at $t \ll 1$:

$$\rho(0) = \frac{2n_s}{\Delta}\left(1 - \frac{1}{\tau_{s0}\Delta}\frac{(1+\alpha)^2}{(1-\alpha)^2}\right)^{-1}\mathcal{P}_\alpha(1,t). \qquad (10)$$

This result is non-perturbative in both t and α. At $t \ll 1$ and $\alpha < 1$, the ratio $\rho(0)/2\nu$ is small due to the exponential dependence of $\mathcal{P}_\alpha(1,t)$ on $1/t$. [The factor in the brackets in Eq. (10) diverges due to the gap closing ($E_g \to 0$) as $\tau_{s0}\Delta \to (\frac{1+\alpha}{1-\alpha})^2$.] The dependence of $\rho(0)$ on t for some values of α is shown in the inset to Fig. 4. Its non-monotonicity is related to that of $\mathcal{P}_\alpha(1,t)$ as a function of t. At a fixed value of t, $\rho(0)$ behaves non-monotonically with the impurity strength α at a given value of $1/(\tau_{s0}\Delta)$.

Discussions. — Our main Eq. (3) could be derived for a toy model of Poissonian magnetic impurities with the strength independently distributed according to $\mathcal{P}_\alpha(\mathsf{a},t)$. We emphasize however that in a disordered film the log-normal distribution is degenerated intrinsically due to mesoscopic fluctuations of the local DOS.

The log-normal distribution $\mathcal{P}_\alpha(\mathsf{a},t)$ predicts an exponentially small probability for realization of very small and very large values of the effective impurity strength a. As well-known from the theory of mesoscopic fluctuations of the local DOS and wave function multifractality, this implies that typically the impurity strength $\mathsf{a}_- < \mathsf{a} < \mathsf{a}_+$ is realized [46]. Using results of Ref. [47], we obtain the following estimate for the termination points: $\mathsf{a}_\pm = \alpha\exp[\pm(4/\sqrt{\pi g})\ln 1/(T_c\tau)]$ [40]. In order our result (10) for $\rho(0)$ were applicable to a typical sample, vicinity of $\mathsf{a} = 1$ should be inside the interval $(\mathsf{a}_-, \mathsf{a}_+)$. It is fulfilled provided $(4/\sqrt{\pi g})\ln 1/(T_c\tau) \gg \ln(1/\alpha)$.

We emphasize the difference with Ref. [35], where mesoscopic fluctuations were not taken into account: (i) in our approach, the DOS is modified already at the mean-field level, and (ii) our result (10) is parametrically larger than the instanton tail of the DOS [31, 35]. The latter is determined by the sheet resistance $1/g$ and is proportional to $\mathcal{P}_\alpha(1, 2/\pi g)$, while our result $\mathcal{P}_\alpha(1,t)$ involves a much larger *spreading resistance* $\ln(\xi/l)/2\pi g$.

Although our results were derived for a weak disorder, $t \ll 1$, they can be extended to the case of a moderate disorder, $t \sim 1$ (provided $g \gg 1$) [45]. In this situation the mean-field equation for θ_ε remains the same as Eq. (3), but the distribution function $\mathcal{P}_\alpha(\mathsf{a},t)$ must be found taking Fermi-liquid renormalizations into account.

The enhancement of magnetic disorder due to mesoscopic fluctuations is not restricted to classical magnetic impurities. It is known [48–51] that the Kondo effect in the disordered electron systems is also modified by mesoscopic fluctuations of the local DOS. Therefore, the theory for the interplay of the Kondo effect and superconductivity developed in Refs. [22–24] needs to be modified for disordered films [45].

The dependence of T_c on the film conductance can be caused by a variety of reasons, among which are the dependence of the DOS at the Fermi energy on disorder, renormalization of the Cooper channel attraction in ballistic and diffusive regimes, Berezinskii-Kosterlitz-Thouless transition etc. [41, 42] The sensitivity of the spin-flip rate on the conductance is a new mechanism providing a nontrivial dependence of T_c on g.

Conclusions. — To summarize, we reconsidered the problem of rare classical magnetic impurities with the Poissonian statistics in a dirty superconducting film. We took into account renormalization of the multiple spin-flip scattering due to mesoscopic fluctuations of the local DOS in a potential disorder. This effect results in the log-normal distribution of the effective magnetic impurity strength rendering the energies of quasiparticle bound states position dependent (see Fig. 1). In the superconducting state, this results in the smearing of the hard gap (obtained in the absence of spin-flip renormalization) and emergence of a non-zero DOS for all energies already at the mean-field level. Depending on the bare magnetic impurity strength, the superconducting transition temperature is suppressed by the spin-flip scattering slower or faster than in the absence of renormalization. Finally, we mention that our results can be extended to the model with an arbitrary distribution of magnetic impurities, the vicinity of a superconductor-insulator transition, the case with Coulomb repulsion in addition to attraction, the presence of Zeeman splitting, etc. [45].

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ONLINE SUPPORTING INFORMATION

Magnetic disorder in superconductors: Enhancement by mesoscopic fluctuations

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In this notes we present details of the derivation of (i) the mean-field equation for the spectral angle, (ii) the results for the spin-flip rate, and (iii) the analytical solution for the density of states.

I. NONLINEAR SIGMA MODEL FOR PARAMAGNETIC IMPURITIES

The low energy description of a two-dimensional disordered superconductor with rare paramagnetic impurities is given by the following nonlinear sigma-model action

$$\mathcal{S} = \mathcal{S}_D + \mathcal{S}_\Delta + \mathcal{S}_{\rm mag}. \tag{S1}$$

Here $\mathcal{S}_D$ is the standard diffusive action (see Refs. [1, 2] for a review):

$$\mathcal{S}_D = \frac{\pi\nu}{8}\int d^2\boldsymbol{r}\,\mathrm{tr}\big[D(\nabla Q)^2 - 4(\varepsilon\tau_3+\Delta\tau_1)Q\big], \tag{S2}$$

where ν and D denote the density of states at the Fermi energy (per one spin projection) and the diffusive coefficient in the normal state, respectively. The matrix Q operates in the spin, Nambu, replica, and Matsubara energy spaces. It is subject to the following constraints [3]:

$$Q^2 = 1, \qquad Q = \overline{Q} \equiv \tau_1\sigma_2 Q^{\mathsf{T}}\tau_1\sigma_2. \tag{S3}$$

Here the transposition $^{\mathsf{T}}$ acts in both the Matsubara energy space and the replica space. The Pauli matrices τ_j (σ_j) act in the Nambu (spin) spaces. The matrix ε is the diagonal matrix with the elements $\varepsilon_n = \pi T(2n+1)$.

The superconducting correlations are described by the order-parameter matrix Δ which is diagonal in the Nambu space with matrix elements $\Delta^a(\boldsymbol{r})$. In the absence of a supercurrent, Δ is chosen to be real. The action $\mathcal{S}_\Delta$ reads

$$\mathcal{S}_\Delta = \frac{\nu}{\lambda T}\int d^2\boldsymbol{r}\sum_{a=1}^{N}|\Delta^a(\boldsymbol{r})|^2. \tag{S4}$$

Here N stands for the number of replica and $\lambda > 0$ denotes the attraction amplitude in the Cooper-channel.

We consider the case of rare classical magnetic impurities with the concentration n_s [the precise condition on n_s is given by Eq. (S18)], when the magnetic part of the action, $\mathcal{S}_{\rm mag}$, becomes separable in the individual magnetic impurities [4]:

$$\mathcal{S}_{\rm mag} \approx \sum_j s^{(j)}_{\rm mag} = -\frac{1}{2}\sum_j \mathrm{tr}\ln\left(1 + i\sqrt{\alpha}\,Q(\boldsymbol{r}_j)\tau_3\boldsymbol{\sigma}\boldsymbol{n}_j\right). \tag{S5}$$

Here $\boldsymbol{n}_j$ stands for the three-dimensional unit vector and the dimensionless parameter $\alpha = (\pi\nu JS)^2$ is expressed in terms of the impurity spin S and exchange constant J. We note that approximation (S5) of the full action $\mathcal{S}_{\rm mag}$ is equivalent to the self-consistent T-matrix approximation for magnetic scattering which treats all orders in scattering off a single magnetic impurity but neglects diagrams with intersecting impurity lines.

Performing the Poisson averaging over positions of the magnetic impurities with the help of the following relation [5]

$$\Big\langle \exp\sum_j f(\boldsymbol{r_j})\Big\rangle = \exp\left\{n_s\int d^2\boldsymbol{r}\left[e^{f(\boldsymbol{r})}-1\right]\right\}, \tag{S6}$$

we find that the contribution to the nonlinear sigma model action due to magnetic impurities becomes

$$\mathcal{S}_{\rm mag} \to -n_s\int d^2\boldsymbol{r}\left(\left\langle\exp\left[\frac{1}{2}\mathrm{tr}\ln\left(1+i\sqrt{\alpha}\,Q(\boldsymbol{r})\tau_3\boldsymbol{\sigma}\boldsymbol{n}\right)\right]\right\rangle_{\boldsymbol{n}} - 1\right). \tag{S7}$$

Here $\langle \dots \rangle_{\boldsymbol{n}}$ stands for the averaging over direction of the unit vector $\boldsymbol{n}$. Expanding $\mathcal{S}_{\rm mag}$ in powers of $\sqrt{\alpha}$, we find

$$\mathcal{S}_{\rm mag} = -n_s \int d^2\boldsymbol{r} \left[\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m} \hat{C} T_m + \frac{1}{2!} \sum_{m,n=1}^{\infty} \frac{(-1)^{m+n}}{4mn} \hat{C} T_{mn} + \frac{1}{3!} \sum_{m,n,p=1}^{\infty} \frac{(-1)^{m+n+p-1}}{8mnp} \hat{C} T_{mnp} + \dots \right]. \tag{S8}$$

Here we introduced the operators

$$\hat{C}T_m = C_{i_1\dots i_m} \operatorname{tr}(QA_{i_1}\dots QA_{i_m}), \qquad \hat{C}T_{mn} = C_{i_1\dots i_{m+n}} \operatorname{tr}(QA_{i_1}\dots QA_{i_m}) \operatorname{tr}(QA_{i_{m+1}}\dots QA_{i_{m+n}}), \tag{S9}$$

and so on. The operator $\hat{C}$ acts as the symmetric tensor:

$$C_{i_1\dots i_m} = \langle n_{i_1} \dots n_{i_m} \rangle. \tag{S10}$$

For convenience we defined the self-dual matrix

$$\boldsymbol{A} = i\sqrt{\alpha}\tau_3 \boldsymbol{\sigma} = \overline{\boldsymbol{A}}. \tag{S11}$$

Since operators $T_{n\dots}$ are symmetric with respect to its indices, the expansion can be written in the following form:

$$\mathcal{S}_{\rm mag} = n_s \int d^2\boldsymbol{r} \left[\frac{1}{4}\hat{C}T_2 - \frac{1}{8}\hat{C}T_{11} + \frac{1}{8}\hat{C}T_4 - \frac{1}{12}\hat{C}T_{31} - \frac{1}{32}\hat{C}T_{22} + \frac{1}{32}\hat{C}T_{211} - \frac{1}{384}\hat{C}T_{1111} + \dots \right]. \tag{S12}$$

This magnetic part of the action should be subjected to the renormalization. As we shall see below, after the renormalization this part of the action can be written in a general way as

$$\mathcal{S}_{\rm mag} = n_s \int d^2\boldsymbol{r} \left[\gamma_2\hat{C}T_2 + \gamma_{11}\hat{C}T_{11} + \gamma_4\hat{C}T_4 + \gamma_{31}\hat{C}T_{31} + \gamma_{22}\hat{C}T_{22} + \gamma_{211}\hat{C}T_{211} + \gamma_{1111}\hat{C}T_{1111} + \dots \right], \tag{S13}$$

where initial values of the coefficients γ follow from Eq. (S12).

II. RENORMALIZATION OF THE ACTION $\mathcal{S}_{\rm mag}$

To renormalize $\mathcal{S}_{\rm mag}$ we write $Q = \Lambda(1 + W + \dots)$, where W obeys two linear constraints: $\Lambda W + W\Lambda = 0$ and $W = -\overline{W}$, and the convergency condition $W = -W^\dagger$. The matrix Λ is assumed to be self-dual: $\Lambda = \overline{\Lambda}$. Then the quadratic part of the action reads:

$$\mathcal{S}_D^{(2)}[W] = -\frac{\pi\nu D}{8} \int d\mathbf{r} \operatorname{tr}(\nabla W)^2. \tag{S14}$$

The quadratic part of the action determines the following contraction rules:

$$\partial_t \langle \operatorname{tr} AW \operatorname{tr} BW \rangle = \operatorname{tr} \left[AB - A\Lambda B\Lambda + A\overline{B} - A\Lambda\overline{B}\Lambda \right], \tag{S15a}$$

$$\partial_t \langle \operatorname{tr} AWBW \rangle = [\operatorname{tr} A \operatorname{tr} B - \operatorname{tr} A\Lambda \operatorname{tr} B\Lambda] - \operatorname{tr} \left[A\overline{B} - A\Lambda\overline{B}\Lambda \right], \tag{S15b}$$

where $t = [2/(\pi g)] \ln(L/l)$. Here $g = 4\pi\nu D$ stands for the sheet conductance and L denotes the infrared length scale.

Next we write the matrix Q as $Q = U^{-1}\Lambda(1 + W + \dots)U$ where the slow field U obeys the condition $\overline{U} = U^{-1}$. Using contraction rules (S15), we find

$$\partial_t \operatorname{tr} AQBQ = \partial_t \langle \operatorname{tr} UAU^{-1}\Lambda WUBU^{-1}\Lambda W \rangle = [\operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AQ \operatorname{tr} BQ] - \operatorname{tr} \left[A\overline{B} - AQ\overline{B}Q \right], \tag{S16a}$$

$$\partial_t \operatorname{tr} AQ \operatorname{tr} BQ = \partial_t \langle \operatorname{tr} UAU^{-1}\Lambda W \operatorname{tr} UBU^{-1}\Lambda W \rangle = -\operatorname{tr} \left[AQBQ - AB - A\overline{B} + AQ\overline{B}Q \right]. \tag{S16b}$$

The action for the slow modes after integration over fast modes W can be found as

$$\mathcal{S}_{\rm mag} \to -\ln \langle e^{-\mathcal{S}_{\rm mag}} \rangle_W = \langle \mathcal{S}_{\rm mag} \rangle_W - \langle\langle \mathcal{S}_{\rm mag}^2 \rangle\rangle_W / 2 + \dots. \tag{S17}$$

Here $\langle \dots \rangle_W$ denotes the averaging over fast modes W. In what follows, in the expansion in the right-hand side of Eq. (S17) we neglect all terms except the lowest order one in the impurity concentration, $\langle \mathcal{S}_{\rm mag} \rangle_W$. The smallness of the omitted terms is controlled by the condition

$$n_s \xi^2 / g \ll 1, \tag{S18}$$

where $\xi \sim \sqrt{D/T_c}$ is the superconducting coherence length in the dirty limit. As we shall see below, it is the first term in the right-hand side of Eq. (S17) that is responsible for the logarithmic renormalization of $\mathcal{S}_{\rm mag}$.

A. Operators of the second order in Q

In the Born approximation (first order in α) we need to consider the operators T_2 and T_{11} with two Q matrices involved. Their contribution to $\mathcal{S}_{\rm mag}$ is controlled by the coefficients γ_2 and γ_{11} with the initial conditions following from (S12): $\gamma_2(0) = 1/4$ and $\gamma_{11}(0) = -1/8$.

Using the contraction rules (S16), we find

$$\partial_t \begin{pmatrix} \hat{C}T_2 \\ \hat{C}T_{11} \end{pmatrix} = M_2 \begin{pmatrix} \hat{C}T_2 \\ \hat{C}T_{11} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}. \tag{S19}$$

The operators T_2 and T_{11} transform into each other under the renormalization. We note that under renormalization the operators with the same or fewer number of Q matrices are generated only. The eigenvalues of M_2 are equal to 2 and -1. The eigenvalue 2 corresponds to the operator $\hat{C}T_2 - \hat{C}T_{11}/2$:

$$\partial_t \left(\hat{C}T_2 - \frac{1}{2}\hat{C}T_{11} \right) = 2 \left(\hat{C}T_2 - \frac{1}{2}\hat{C}T_{11} \right). \tag{S20}$$

The operator $\hat{C}T_2 - \hat{C}T_{11}/2$ is known to be a pure scaling operator beyond the lowest order perturbation theory [6–9].

We emphasize that the operators of the second order in Q enter the magnetic part of the action, Eq. (S12), precisely in combination $\hat{C}T_2 - \hat{C}T_{11}/2$. This implies that

$$\gamma_2(t) = \frac{1}{4}e^{2t}, \qquad \gamma_{11}(t) = -\frac{1}{8}e^{2t}. \tag{S21}$$

B. Operators of the fourth order in Q

The next nontrivial order in α involves operators which are of the fourth order in Q. Their flow is described by the system

$$\partial_t \hat{C}T_4 = 6(\hat{C}T_4) - 4(\hat{C}T_{31}) - 2(\hat{C}T_{22}) - 4\alpha\hat{C}T_2, \tag{S22a}$$

$$\partial_t \hat{C}T_{31} = -6(\hat{C}T_4) + 3(\hat{C}T_{31}) - 3(\hat{C}T_{211}) - 6\alpha\hat{C}T_2 - 3\alpha\hat{C}T_{11}, \tag{S22b}$$

$$\partial_t \hat{C}T_{22} = -8(\hat{C}T_4) + 2(\hat{C}T_{22}) - 2(\hat{C}T_{211}), \tag{S22c}$$

$$\partial_t \hat{C}T_{211} = -8(\hat{C}T_{31}) - 2(\hat{C}T_{22}) + (\hat{C}T_{211}) - (\hat{C}T_{1111}) - 8\alpha\hat{C}T_{11}, \tag{S22d}$$

$$\partial_t \hat{C}T_{1111} = -12(\hat{C}T_{211}). \tag{S22e}$$

The operators of the forth order in Q are mixed under the renormalization. In addition, the operators of the second order in Q are generated. The system of equations (S22) can be cast in the matrix form

$$\partial_t \begin{pmatrix} \hat{C}T_4 \\ \hat{C}T_{31} \\ \hat{C}T_{2,2} \\ \hat{C}T_{211} \\ \hat{C}T_{1111} \\ \hat{C}T_2 \\ \hat{C}T_{11} \end{pmatrix} = M_4 \begin{pmatrix} \hat{C}T_4 \\ \hat{C}T_{31} \\ \hat{C}T_{2,2} \\ \hat{C}T_{211} \\ \hat{C}T_{1111} \\ \hat{C}T_2 \\ \hat{C}T_{11} \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 6 & -4 & -2 & 0 & 0 & -4\alpha & 0 \\ -6 & 3 & 0 & -3 & 0 & -6\alpha & -3\alpha \\ -8 & 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & -8 & -2 & 1 & -1 & 0 & -8\alpha \\ 0 & 0 & 0 & -12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}. \tag{S23}$$

Here we used Eq. (S19). We emphasize that the matrix M_4 is the upper triangular block matrix. This reflects the fact that under renormalization the operators with the same or fewer number of Q matrices are generated only. The matrix M_4 has the following eigenvalues: 12, 5, 2, 2, -1, -1, and -6. The largest eigenvalue 12 corresponds to the operator $\hat{C}T_4 - (2/3)\hat{C}T_{31} - (1/4)\hat{C}T_{22} + (1/4)\hat{C}T_{211} - (1/48)\hat{C}T_{1111}$. It is known that this operator is the pure scaling operator from arguments based on the group representation theory [6–9]. It is worth emphasizing that the operators of the forth order in Q enter the magnetic part of the action, Eq. (S12), precisely in the combination $\hat{C}T_4 - (2/3)\hat{C}T_{31} - (1/4)\hat{C}T_{22} + (1/4)\hat{C}T_{211} - (1/48)\hat{C}T_{1111}$. This implies that the coefficients in the action (S13) are simply

$$\gamma_4(t) = \frac{1}{8}e^{12t}, \quad \gamma_{31}(t) = -\frac{1}{12}e^{12t}, \quad \gamma_{22}(t) = -\frac{1}{32}e^{12t}, \quad \gamma_{211}(t) = \frac{1}{32}e^{12t}, \quad \gamma_{1111}(t) = -\frac{1}{384}e^{12t}. \tag{S24}$$

C. Renormalization of operators of arbitrary order in Q

In general, one can derive the following set of renormalization group equations:

$$\partial_t \hat{C}T_n = \frac{n(n-1)}{2}\hat{C}T_n - \frac{n}{2}\sum_{k=1}^{n-1}\Bigl(\hat{C}T_{k,n-k} - (-\alpha)^{\min(k,n-k)}\hat{C}T_{|n-2k|}\Bigr), \tag{S25}$$

$$\begin{aligned}\partial_t \hat{C}T_{m,n} = {} & -2mn\Bigl(\hat{C}T_{m+n} - (-\alpha)^{\min(m,n)}\hat{C}T_{|m-n|}\Bigr) + \frac{m(m-1)+n(n-1)}{2}\hat{C}T_{m,n} - \Biggl[\frac{m}{2}\sum_{k=1}^{m-1}\Bigl(\hat{C}T_{k,m-k,n} \\ & - (-\alpha)^{\min(k,m-k)}\hat{C}T_{|m-2k|,n}\Bigr) + \frac{n}{2}\sum_{l=1}^{n-1}\Bigl(\hat{C}T_{m,l,n-l} - (-\alpha)^{\min(l,n-l)}\hat{C}T_{m,|n-2l|}\Bigr)\Biggr],\end{aligned} \tag{S26}$$

$$\begin{aligned}\partial_t \hat{C}T_{m,n,p} = {} & -2\Bigl[mn\Bigl(\hat{C}T_{m+n,p} - (-\alpha)^{\min(m,n)}\hat{C}T_{|m-n|,p}\Bigr) + mp\Bigl(\hat{C}T_{m+p,n} - (-\alpha)^{\min(m,p)}\hat{C}T_{|m-p|,n}\Bigr) \\ & + np\Bigl(\hat{C}T_{m,n+p} - (-\alpha)^{\min(n,p)}\hat{C}T_{m,|n-p|}\Bigr)\Bigr] + \frac{m(m-1)+n(n-1)+p(p-1)}{2}(\hat{C}T_{m,n,p}) \\ & - \Biggl[\frac{m}{2}\sum_{k=1}^{m-1}\Bigl(\hat{C}T_{k,m-k,n,p} - (-\alpha)^{\min(k,m-k)}\hat{C}T_{|m-2k|,n,p}\Bigr) + \frac{n}{2}\sum_{l=1}^{n-1}\Bigl(\hat{C}T_{m,l,n-l,p} \\ & - (-\alpha)^{\min(l,n-l)}\hat{C}T_{m,|n-2l|,p}\Bigr) + \frac{p}{2}\sum_{s=1}^{p-1}\Bigl(\hat{C}T_{m,n,s,p-s} - (-\alpha)^{\min(s,p-s)}\hat{C}T_{m,n,|p-2s|}\Bigr)\Biggr].\end{aligned} \tag{S27}$$

and so on. Using these equations we find for the renormalization of the action:

$$\begin{aligned}\partial_t \mathcal{S}_{\text{mag}} = {} & -n_s\int d^2\boldsymbol{r}\Biggl\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m}\Bigl[\frac{m(m-1)}{2}\hat{C}T_m + \frac{m}{2}\sum_{k=1}^{m-1}(-\alpha)^{\min(k,m-k)}\hat{C}T_{|m-2k|}\Bigr] \\ & + \frac{1}{2!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{4mn}(-2mn)\Bigl[\hat{C}T_{m+n} - (-\alpha)^{\min(m,n)}\hat{C}T_{|m-n|}\Bigr] + \sum_{m=1}^{\infty}\frac{(-1)^m}{2m}\frac{m}{2}\sum_{k=1}^{m-1}\hat{C}T_{k,m-k} \\ & + \frac{1}{2!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{4mn}\Bigl[m(m-1)\hat{C}T_{m,n} + m\sum_{k=1}^{m-1}(-\alpha)^{\min(k,m-k)}\hat{C}T_{|m-2k|,n}\Bigr] \\ & + \frac{1}{3!}\sum_{m,n,p=1}^{\infty}\frac{(-1)^{m+n+p-1}}{8mnp}(-6mn)\Bigl[\hat{C}T_{m+n,p} - (-\alpha)^{\min(m,n)}\hat{C}T_{|m-n|,p}\Bigr] + \dots\Biggr\} \\ = {} & -n_s\int d^2\boldsymbol{r}\Biggl\{\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m}m(m-1)\hat{C}T_m + \frac{1}{2!}\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{4mn}(m+n)(m+n-1)\hat{C}T_{mn} + \dots\Biggr\}\end{aligned} \tag{S28}$$

Note that all terms in Eq. (S28) which contain α cancel each other.

D. The renormalized action $\mathcal{S}_{\text{mag}}$

All in all, we find from Eq. (S28) that the coefficients $\gamma_{k_1 k_2 \dots k_q}$, where $k_1 + k_2 + \dots + k_q = n$, behaves in the same way:

$$\gamma_{k_1 k_2 \dots k_q}(t) = \gamma_{k_1 k_2 \dots k_q}(0)e^{n(n-1)t}. \tag{S29}$$

In what follows we are interested in the mean-field analysis of the renormalized action (S13) for which the singlet sector of the theory is important only. Therefore, one can operate with Q matrix which is the unit matrix in the spin space, $Q = Q_0\sigma_0$. Then averaging over directions of the impurity magnetization $\boldsymbol{n}$ becomes trivial. We find (all indices, m, n, … are even)

$$\hat{C}T_m = (-\alpha)^{m/2}\operatorname{tr}(Q\tau_3)^m, \qquad \hat{C}T_{mn} = (-\alpha)^{(m+n)/2}\operatorname{tr}(Q\tau_3)^m \operatorname{tr}(Q\tau_3)^n, \tag{S30}$$

and so on. Then the renormalized action for magnetic impurities becomes

$$\mathcal{S}_{\rm mag} = -n_s\int d^2\boldsymbol{r}\Bigg[-\sum_{k=1}^{\infty}\frac{(-\alpha)^k}{2^2k}e^{2k(2k-1)t}\operatorname{tr}(Q\tau_3)^{2k} + \frac{1}{2!}\sum_{k,l=1}^{\infty}\frac{(-\alpha)^{k+l}}{4^2kl}e^{(2k+2l)(2k+2l-1)t}\operatorname{tr}(Q\tau_3)^{2k}\operatorname{tr}(Q\tau_3)^{2l} \\ - \frac{1}{3!}\sum_{k,l,m=1}^{\infty}\frac{(-\alpha)^{k+l+m}}{8^2klm}e^{(2k+2l+2m)(2k+2l+2m-1)t}\operatorname{tr}(Q\tau_3)^{2k}\operatorname{tr}(Q\tau_3)^{2l}\operatorname{tr}(Q\tau_3)^{2m} + \dots\Bigg]. \tag{S31}$$

Decoupling the Gaussian part with an auxiliary integral over λ we obtain

$$\mathcal{S}_{\rm mag} = -\overline{n}_s\int d^2\boldsymbol{r}\int\frac{d\lambda}{\sqrt{4\pi t}}e^{-(\lambda+t)^2/4t}\Bigg[-\sum_{k=1}^{\infty}\frac{(-\alpha)^k}{2^2k}e^{2k\lambda}\operatorname{tr}(Q\tau_3)^{2k} + \frac{1}{2!}\sum_{k,l=1}^{\infty}\frac{(-\alpha)^{k+l}}{4^2kl}e^{(2k+2l)\lambda}\operatorname{tr}(Q\tau_3)^{2k}\operatorname{tr}(Q\tau_3)^{2l} \\ - \frac{1}{3!}\sum_{k,l,m=1}^{\infty}\frac{(-\alpha)^{k+l+m}}{8^2klm}e^{(2k+2l+2m)\lambda}\operatorname{tr}(Q\tau_3)^{2k}\operatorname{tr}(Q\tau_3)^{2l}\operatorname{tr}(Q\tau_3)^{2m} + \dots\Bigg]. \tag{S32}$$

Now all summations become trivial:

$$\mathcal{S}_{\rm mag} = -n_s\int d^2\boldsymbol{r}\int\frac{d\lambda}{\sqrt{4\pi t}}e^{-(\lambda+t)^2/4t}\left[X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots\right] = -n_s\int d^2\boldsymbol{r}\int\frac{d\lambda}{\sqrt{4\pi t}}e^{-(\lambda+t)^2/4t}(e^X - 1), \tag{S33}$$

where

$$X = -\operatorname{tr}\sum_{k=1}^{\infty}\frac{(-\alpha)^k}{4k}e^{2k\lambda}(Q\tau_3)^{2k} = \frac{1}{4}\operatorname{tr}\ln[1+\alpha e^{2\lambda}(Q\tau_3)^2]. \tag{S34}$$

Finally, we find

$$\mathcal{S}_{\rm mag} = -n_s\int d^2\boldsymbol{r}\int\frac{d\lambda}{\sqrt{4\pi t}}\exp\left\{-\frac{(\lambda+t)^2}{4t}\right\}\left[\exp\left\{\frac{1}{4}\operatorname{tr}\ln[1+\alpha e^{2\lambda}(Q\tau_3)^2]\right\} - 1\right], \tag{S35}$$

where tr still includes summation over the spin space. It is convenient to introduce the following short-hand notation:

$$\mathcal{S}_{\rm mag} = -n_s\int d^2\boldsymbol{r}\left\langle\exp\left\{\frac{1}{4}\operatorname{tr}\ln[1+\mathsf{a}(Q\tau_3)^2]\right\} - 1\right\rangle_{\mathsf{a}} \tag{S36}$$

where

$$\langle\cdots\rangle_{\mathsf{a}} = \int_0^{\infty} d\mathsf{a}\,\mathcal{P}_\alpha(\mathsf{a},t)\dots, \qquad \mathcal{P}_\alpha(\mathsf{a},t) = \frac{1}{4\mathsf{a}\sqrt{\pi t}}\exp\left[-\frac{1}{4t}\left(\frac{1}{2}\ln\frac{\mathsf{a}}{\alpha} + t\right)^2\right]. \tag{S37}$$

Comparing (S36) with (S7) we may interpret the effect of renormalization as follows. Now instead of a single value of α there is a log-normal distribution of the effective strength of impurity a.

For the study of space-averaged configurations at the mean-field level, it is sufficient to retain the term of the first order in trace only in the renormalized action (S36):

$$\mathcal{S}_{\rm mag}^{\rm MF} = -\frac{n_s}{4}\int d^2\boldsymbol{r}\left\langle\operatorname{tr}\ln[1+\mathsf{a}(Q\tau_3)^2]\right\rangle_{\mathsf{a}}. \tag{S38}$$

For a superconducting system, the mean-field solution can be parametrized as

$$Q = \tau_1\sin\theta + \tau_3\cos\theta. \tag{S39}$$

Since the eigenvalues of $(Q\tau_3)^2$ are $e^{\pm 2i\theta}$, we find

$$\mathcal{S}_{\rm mag}^{\rm MF} = -\frac{n_s}{2}\int d^2\boldsymbol{r}\left\langle\left[\ln\big(1+\mathsf{a}\,e^{2i\theta}\big) + \ln\big(1+\mathsf{a}\,e^{-2i\theta}\big)\right]\right\rangle_{\mathsf{a}}. \tag{S40}$$

III. THE MEAN-FIELD EQUATIONS, THE SPIN-FLIP RATE, AND THE DENSITY OF STATES

Performing variation of the action $\mathcal{S}$ [with $\mathcal{S}_{\mathrm{mag}}$ given by Eq. (S38)] on the configuration (S39) with respect to Δ and θ_ε, we find the following mean-field equations: (i) the self-consistency equation,

$$\Delta = \pi\lambda T \sum_\varepsilon \sin\theta_\varepsilon, \tag{S41}$$

and (ii) the Usadel equation,

$$\varepsilon \sin\theta_\varepsilon - \Delta\cos\theta_\varepsilon + \frac{n_s}{\pi\nu}\left\langle \frac{\mathsf{a}\sin 2\theta_\varepsilon}{1+\mathsf{a}^2+2\mathsf{a}\cos 2\theta_\varepsilon}\right\rangle_{\mathsf{a}} = 0. \tag{S42}$$

A. The spin-flip rate near the transition temperature

At $T\to T_c$ one can linearize Eq. (S42) in θ_ε, which yields $\theta_\varepsilon = \Delta/(\varepsilon+1/\tau_s)$, where the spin-flip rate is given by

$$\frac{1}{\tau_s} = \frac{2n_s}{\pi\nu}\left\langle\frac{\mathsf{a}}{(1+\mathsf{a})^2}\right\rangle_{\mathsf{a}} = \frac{2n_s}{\pi\nu}\int\limits_{-\infty}^{\infty}\frac{du}{\sqrt{\pi}}e^{-u^2-2i\mu u}\int\limits_0^\infty \frac{d\lambda}{\sqrt{\pi}}\frac{\cos(2u\lambda)}{\cosh(\beta\lambda)+1} = \frac{2n_s}{\pi\nu}\frac{4}{\beta^2}\int\limits_{-\infty}^{\infty} du\, e^{-u^2-2iu\mu}\frac{u}{\sinh\frac{2\pi u}{\beta}}. \tag{S43}$$

Here we introduced $\mu = (2t-\ln\alpha)/(4\sqrt{t})$ and $\beta = 4\sqrt{t}$. Expanding the denominator in the last integral in the right hand side of Eq. (S43) in powers of $\exp(-2\pi|u|/\beta)$ and, then, performing integration over u, we find

$$\frac{1}{\tau_s} = -\frac{2n_s}{\pi\nu}\frac{2\sqrt{\pi}}{\beta^2}\partial_\mu \operatorname{Im}\sum_{k=0}^{\infty} f\big(i\mu+\pi(2k+1)/\beta\big). \tag{S44}$$

Here we introduce the function $f(z) = \exp(z^2)[1-\operatorname{erf}(z)]$. At $\beta/\pi \ll 1$ we can use the expansion of the function $f(z)$ in series in $1/z$:

$$f(z) = \sum_{l=1}^{\infty}\frac{(-1)^{l-1}\Gamma(l-1/2)}{\pi z^{2l-1}}. \tag{S45}$$

Performing summation over k in (S44), we obtain

$$\frac{1}{\tau_s} = \frac{n_s}{\pi\nu\beta}\sum_{l=0}^{\infty}\frac{1}{4^l l!}\partial_\mu^{2l+1}\tan\left(\frac{\beta\mu}{2}\right) = \frac{2n_s}{\pi\nu}e^{4t(\alpha\partial_\alpha)^2}\frac{\alpha e^{-2t}}{(1+\alpha e^{-2t})^2}. \tag{S46}$$

In the limiting cases Eq. (S46) reduces to

$$\frac{1}{\tau_s} = \frac{2n_s\alpha}{\pi\nu}\begin{cases}\alpha e^{2t}, & \alpha\to 0,\\ e^{6t}/\alpha, & \alpha\to\infty.\end{cases} \tag{S47}$$

Expanding the right hand side of Eq. (S46) to the first order in t, we find

$$\frac{1}{\tau_s} = \frac{1}{\tau_{s0}}\left[1+2t\frac{1-8\alpha+3\alpha^2}{(1+\alpha)^2}+O(t^2)\right], \tag{S48}$$

where $1/\tau_{s0} = 2\alpha n_s/[\pi\nu(1+\alpha)^2]$.

For $\beta/\pi \gg 1$, the sum k in Eq. (S44) reduces to the integral. Then, we find

$$\frac{1}{\tau_s} = \frac{2n_s}{\pi\nu}\frac{1}{\sqrt{\pi}\beta}\operatorname{Re} f\big(i\mu+\pi/\beta\big) \tag{S49}$$

Next, for $\mu\beta \gg 1$, which holds for $t \gg 1$, we obtain

$$\frac{1}{\tau_s} = \frac{2n_s}{\pi\nu}\frac{1}{\sqrt{\pi}\beta}e^{-\mu^2} = \frac{1}{\tau_{s0}}\frac{(1+\alpha)^2}{\alpha}\mathcal{P}_\alpha(1,t) \tag{S50}$$

B. The density of states

The average DOS can be extracted from $\langle Q_{\varepsilon\varepsilon}\rangle$ analytically continued to the real energies E: $i\varepsilon \to E + i0^+$. The mean-field equation (S42) can be written as

$$\varepsilon \sin\theta_\varepsilon - \Delta\cos\theta_\varepsilon + \frac{n_s}{\pi\nu}\mathcal{F}\big(\theta_\varepsilon, (2t-\ln\alpha)/(4\sqrt{t}), 4\sqrt{t}\big) = 0, \tag{S51}$$

where

$$\mathcal{F}(\theta,\mu,\beta) = \int\limits_{-\infty}^{\infty} \frac{d\lambda}{2\sqrt{\pi}} e^{-(\lambda+\mu)^2} \frac{\sin 2\theta}{\cosh(\beta\lambda)+\cos 2\theta}. \tag{S52}$$

It is convenient to parametrize the spectral angle as $\theta_\varepsilon = \pi/2 + i\psi$ such that the density of states becomes:

$$\rho(E) = 2\nu \lim_{i\varepsilon\to E+i0^+} \operatorname{Im}\sinh\psi. \tag{S53}$$

For arbitrary values of t and α, Eq. (S51) is a complicated integral equation which can be solved numerically. Below, we demonstrate how its solution and, consequently, the density of states, can be found analytically at $t \ll 1$.

At first, we rewrite the function $\mathcal{F}(\theta,\mu,\beta)$ as follows

$$\mathcal{F}(\theta,\mu,\beta) = \int\limits_{-\infty}^{\infty} \frac{du}{\sqrt{\pi}} e^{-u^2-2i\mu u} \int\limits_{0}^{\infty} \frac{d\lambda}{\sqrt{\pi}} \frac{\sin 2\theta \cos(2u\lambda)}{\cosh(\beta\lambda)+\cos 2\theta} = \frac{1}{\beta}\int\limits_{-\infty}^{\infty} du\, \frac{\sinh\frac{4\theta u}{\beta}}{\sinh\frac{2\pi u}{\beta}}\, e^{-u^2-2iu\mu}. \tag{S54}$$

Here we used the relation 3.514.2 from the book [10]. Expanding the denominator in the last integral in the right hand side of Eq. (S54) in powers of $\exp(-2\pi|u|/\beta)$ and, then, performing integration over u, we find

$$\mathcal{F}(\pi/2+i\psi,\mu,\beta) = \frac{\sqrt{\pi}}{2\beta}\sum_{\sigma=\pm}\Bigg\{\sum_{k=0}^{\infty} f\big(i\sigma\mu + 2(\pi k - i\psi)/\beta\big) - \sum_{k=1}^{\infty} f\big(i\sigma\mu + 2(\pi k + i\psi)/\beta\big)\Bigg\}, \tag{S55}$$

where $f(z) = \exp(z^2)[1-\operatorname{erf}(z)]$. Since we are interested in $\beta/\pi \lesssim 1$ we can use expansion of the function $f(z)$ in powers of $1/z$ (see Eq. (S45)). Then performing summation over k in the right hand side of Eq. (S55), we find

$$\mathcal{F}(\pi/2+i\psi,\mu,\beta) = \frac{1}{2\beta}\Big[H(\mu-2\psi/\beta) + H(-\mu-2\psi/\beta)\Big] - \frac{1}{2}e^{\frac{1}{4}\partial_\mu^2}\frac{i\sinh(2\psi)}{\cosh(\beta\mu)-\cosh(2\psi)}, \tag{S56}$$

where

$$H(z) = \sqrt{\pi}e^{-z^2}\big[1 - i\operatorname{erfi}(z)\big] + i\, e^{\frac{1}{4}\partial_z^2}\, z^{-1}. \tag{S57}$$

While deriving Eq. (S56) we used the following relation for the Euler di-gamma functions:

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \frac{\pi}{\tan\pi z}. \tag{S58}$$

Using the result (S56) and making transformation $\varepsilon \to -iE$, we obtain the following form of the mean-field equation (S42):

$$e^{4t(\alpha\partial_\alpha)^2} F_E(\psi,\alpha e^{-2t}) - \frac{in_s}{2\pi\beta\nu\Delta}\Big[H\Big(\frac{2t-\ln\alpha-2\psi}{4\sqrt{t}}\Big) + H\Big(\frac{\ln\alpha-2t-2\psi}{4\sqrt{t}}\Big)\Big] = 0 \tag{S59}$$

where the function

$$F_E(\psi,\alpha) = \sinh\psi - \frac{E}{\Delta}\cosh\psi - \frac{n_s}{\pi\nu\Delta}\frac{\alpha\sinh 2\psi}{1+\alpha^2-2\alpha\cosh 2\psi} \tag{S60}$$

is defined in such a way that the mean-field equation at $t=0$ is given as

$$F_E(\psi,\alpha) = 0. \tag{S61}$$

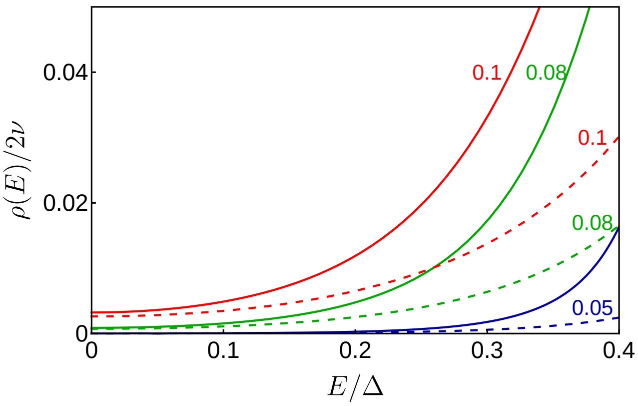

FIG. 1. Energy dependence of the density of states for some values of the parameter t. The solid curves are obtained by numerical solution of the mean-field Eq. (S51). The dashed curves are plotted with the help of Eqs. (S65). We use $1/(\tau_{s0}\Delta) = 0.1$ and $\alpha = 0.05$.

In what follows, we focus at the case

$$\frac{1}{\tau_{s0}\Delta} < \frac{(1-\alpha)}{(1+\alpha)}^2, \tag{S62}$$

in which the density of states has a finite gap E_g at $t = 0$ [11]. In this case, Eq. (S61) has a real solution ψ for $|E| < E_g$.

In order to find the solution of Eq. (S59) for $t \ll 1$, we neglect all regular in t terms. Then Eq. (S59) becomes

$$F_E(\psi, \alpha) = \frac{i n_s}{2\nu\Delta} \sum_{\sigma=\pm} e^{2\psi\sigma} \mathcal{P}_\alpha(e^{2\psi\sigma}, t). \tag{S63}$$

The nonzero right hand side of this equations allows to have complex solutions for ψ even at $|E| < E_g$. Substituting $\psi = \psi' + i\psi''$, assuming that $\psi'' \ll 1$, and splitting Eq. (S63) into real and imaginary parts, we find

$$F_E(\psi', \alpha) = 0, \qquad \partial_{\psi'} F_E(\psi', \alpha)\psi'' = \frac{n_s}{2\nu\Delta} \sum_{\sigma=\pm} e^{2\psi'\sigma} \mathcal{P}_\alpha(e^{2\psi'\sigma}, t) = 0. \tag{S64}$$

The density of states can be found as

$$\rho(E) \approx 2\nu\ \psi'' \cosh\psi' = \frac{n_s}{\Delta} \frac{\cosh\psi'}{\partial_{\psi'} F_E(\psi', t)} \sum_{\sigma=\pm} e^{2\psi'\sigma} \mathcal{P}_\alpha(e^{2\psi'\sigma}, t). \tag{S65}$$

We present the comparison between the density of states found from numerical solution of Eq. (S51) and analytical result (S65) in Fig. 1.

At $E = 0$, $\psi' = 0$ is the solution of Eq. (S61). Then from Eq. (S65) we find the density of states at zero (Fermi) energy

$$\rho(0) = \frac{2n_s}{\Delta}\left(1 - \frac{1}{\tau_{s0}\Delta}\frac{(1+\alpha)^2}{(1-\alpha)^2}\right)^{-1} \mathcal{P}_\alpha(1, t). \tag{S66}$$

IV. THE EFFECT OF TERMINATION OF THE MULTIFRACTAL SPECTRUM

The result (S29) for the coefficients $\gamma_{k_1k_2\dots k_q}$ is derived by consideration of the contributions related with $\langle \mathcal{S}_{\rm mag}\rangle_W$. In this approximation operators $T_{k_1k_2\dots k_q}$ with given $n = k_1 + \dots + k_q$ always transform under the renormalization group into linear combinations of operators $T_{l_1l_2\dots l_q}$ with $m = l_1 + \dots + l_q \leqslant n$. Therefore, the renormalization group equations remain linear in coefficients $\gamma_{k_1k_2\dots k_q}$. In general, one needs to take into account terms which are nonlinear in $\mathcal{S}_{\rm mag}$, e.g. $\langle[\mathcal{S}_{\rm mag}]^2\rangle_W$. Then the fusion of two operators $T_{k_1k_2\dots k_q}$ and $T_{l_1l_2\dots l_q}$ into a single operator $T_{s_1s_2\dots s_q}$ with $s_1+\dots+s_q=n+m$ is possible. This renders the renormalization group equations for $\gamma_{k_1k_2\dots k_q}$ nonlinear [12]. This nonlinearity results in termination of the multifractal spectrum [13] which implies the following modification of Eq. (S29):

$$\gamma_{k_1k_2\dots k_q}(t)=\gamma_{k_1k_2\dots k_q}(0)e^{\mathsf{y}_n t}, \qquad \mathsf{y}_n=\begin{cases} n(n-1), & 1<n<n_c, \\ -n_c^2+(2n_c-1)n, & n_c\leqslant n. \end{cases} \tag{S67}$$

Here $n_c=\sqrt{2/t_0}\gg 1$ and $t_0=2/(\pi g)\ll 1$ denotes the bare resistance. The function y_n obeys the following symmetry property: $\mathsf{y}_{1-n}=\mathsf{y}_n$ [14]. Let us now define the function $\mathcal{G}(\lambda)$ as

$$\int\limits_{-\infty}^{\infty} d\lambda\, e^{n\lambda}\,\mathcal{G}(\lambda)=e^{\mathsf{y}_n t}. \tag{S68}$$

Then we find

$$\mathcal{S}_{\rm mag}=-n_s\int d^2\boldsymbol{r}\int\limits_{-\infty}^{\infty} d\lambda\,\mathcal{G}(\lambda)\bigl(e^X-1\bigr), \tag{S69}$$

where X is given by Eq. (S34).

At $t\ll 1$ and $t/t_0\gg 1$, the function $\mathcal{G}(\lambda)$ can be written as

$$\mathcal{G}(\lambda)=\frac{1}{\sqrt{4\pi t}}\exp\left[-\frac{(\lambda+t)^2}{4t}\right]\theta(\lambda_c-|\lambda|), \tag{S70}$$

where $\theta(z)$ denotes the Heaviside step function and $\lambda_c=t(2\sqrt{2/t_0}-1)\approx 2t\sqrt{2/t_0}$. This form of the function $\mathcal{G}(\lambda)$ implies that the integration over a in Eq. (S42) is restricted to the range $\mathsf{a}_-<\mathsf{a}<\mathsf{a}_+$, where $\mathsf{a}_\pm=\alpha\exp(\pm 2\lambda_c)$. Since for the existence of a finite density of states near the Fermi energy, vicinity of $\mathsf{a}=1$ is important, this point should be within the range of integration over a in Eq. (S42), i.e. $\mathsf{a}_-<1<\mathsf{a}_+$. The latter condition is fulfilled provided $4t\sqrt{2/t_0}\gg\ln(1/\alpha)$.

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